Higgs and Unitarity

The main lesson of these lecture notes is as follows.

- The role of Higgs boson is twofold; Giving mass to the gauge boson, and guaranteeing unitarity of the theory with massive gauge boson.
- Learn what the Goldstone Equivalence theorem is and how to use it.

These points were studied by Lee, Quigg, and Thacker [3]. At that time, the authors theoretically discovered that the Higgs mass (which had not yet been experimentally found) must be lower than about 1 TeV.[1] This paper is also an excellent example demonstrating the practicality of the Goldstone equivalence theorem.

Notation/Glossary

- Metric convention: $\eta_{\mu\nu}$: $(+,-,-,-)$
- Feynman Rule: We use the notation where i is attached to the coupling. The final amplitude is $i\mathcal{M}$.
  - Example: $\mathcal{L} \supset -\frac{\lambda}{4!}\phi^4 \to -i\lambda$
  - For derivative interactions involving ∂_μ, a factor of $(-ip_\mu)$ is attached for ingoing momentum and $(+ip_\mu)$ for outgoing momentum. (Naturally, for ∂^μ, it is $\pm ip^\mu$.)
- COM: Center of momentum (frame)
- $d\Pi$: Phase space integral factor, roughly the product of $\frac{d^3p_i}{2E_i(2\pi)^3}$ terms.

1 Unitarity Bound

The amplitude $\mathcal{M}$ calculated in QFT represents the probability density of a given process occurring. Since this value is related to probability, $\mathcal{M}$ must follow several constraints. The unitarity bound is one such constraint, suggesting that the value of $|\mathcal{M}|$ cannot exceed a certain threshold. Consider a $2 \to 2$ scattering process. In the COM frame, any $2 \to 2$ scattering process can be represented on a two-dimensional plane. Therefore the scattering process itself and its associated physical quantities have rotational invariance: the scattering probability is invariant under rotations about the collision axis. This means that the scattering amplitude $\mathcal{M}$ depends on the scattering angle θ.[2] Thanks to this rotational invariance, $\mathcal{M}$ can be expanded by Legendre Polynomials $P_j(\cos\theta)$.

$$\mathcal{M}(\theta) = 16\pi \sum_{j=0}^{\infty} a_j(2j+1)P_j(\cos\theta) \tag{1}$$

[1]Note that the paper [3] was published in 1977. At that time, even W and Z bosons had not been discovered.

[2]More specifically, it depends only on the Mandelstam variables s and t (which corresponds to the scattering angle).

The scattering cross-section corresponding to the $2 \to 2$ scattering process is as follows.[3]

$$\begin{aligned}
\sigma &= \frac{1}{32\pi s}\int d\cos\theta |\mathcal{M}|^2 = \frac{8\pi}{s}\sum_{j,k=0}^{\infty} a_j a_k^*(2j+1)(2k+1)\int d\cos\theta P_j(\cos\theta)P_k(\cos\theta) \\
&= \frac{16\pi}{s}\sum_{j=0}^{\infty}(2j+1)|a_j|^2
\end{aligned} \tag{2}$$

Now let us turn to the optical theorem. The unitarity of the scattering matrix $\mathcal{S} = 1 + i\mathcal{T} = 1 + i(2\pi)^4\delta^4(\Sigma p)\mathbb{M}$ implies the following:

$$\begin{aligned}
1 = \mathcal{S}\mathcal{S}^\dagger = (1+i\mathcal{T})(1-i\mathcal{T}^\dagger) &= 1 + i(\mathcal{T}-\mathcal{T}^\dagger) + \mathcal{T}\mathcal{T}^\dagger \\
-i(\mathcal{T}-\mathcal{T}^\dagger) = \mathcal{T}\mathcal{T}^\dagger \ : \ (2\pi)^4\delta^4(\Sigma p)(\mathbb{M}-\mathbb{M}^\dagger) &= (2\pi)^8[\delta^4(\Sigma p)]^2\mathbb{M}\mathbb{M}^\dagger \\
-i(\mathbb{M}-\mathbb{M}^\dagger) &= (2\pi)^4\delta^4(\Sigma p)\mathbb{M}\mathbb{M}^\dagger
\end{aligned} \tag{3}$$

Beware that $\mathcal{S}$, $\mathcal{T}$, and $\mathbb{M}$ in this equation are all <u>matrices</u>! They are related to the scattering amplitude $\mathcal{M}$ via $\mathcal{M}(i \to f) = \langle f|\mathbb{M}|i\rangle$. Therefore, Eq. (3) should be interpreted as follows:

$$-i(\langle f|\mathbb{M}|i\rangle - \langle f|\mathbb{M}|i\rangle^*) = \sum_X (2\pi)^4\delta^4(\Sigma p)\langle f|\mathbb{M}|X\rangle\langle X|\mathbb{M}|i\rangle \tag{4}$$

Here X represents all possible final states, so the summation over X can be interpreted as a phase space integral.

$$\sum_X (2\pi)^4\delta^4(\Sigma p)\langle f|\mathbb{M}|X\rangle\langle X|\mathbb{M}|i\rangle \to \sum_X \int d\Pi_X (2\pi)^4\delta^4(\Sigma p)\langle f|\mathbb{M}|X\rangle\langle X|\mathbb{M}|i\rangle \tag{5}$$

Now let us consider a special case: a two-particle initial state and $|i\rangle = |f\rangle$, we can see that Eq. (5) is connected to the cross-section.[4]

$$\sum_X \int d\Pi_X (2\pi)^4\delta^4(\Sigma p)\mathcal{M}(i \to X) = 4p\sqrt{s}\sum_X \int d\sigma(i \to X) = 4p\sqrt{s}\sigma_{\text{tot}} \tag{6}$$

(Here p is the incident momentum in the COM frame.) Also, in this case, $-i\langle i|(\mathbb{M}-\mathbb{M}^\dagger)|i\rangle = 2\text{Im }\mathcal{M}(i \to i)$. By combining Eq. (3) and Eq. (6), we get following optical theorem.

$$\text{Im }\mathcal{M}(i \to i) = 2p\sqrt{s}\sigma_{\text{tot}} \tag{7}$$

Now let's combine the partial wave decomposition and the optical theorem. Substitute Eq. (1) into the left side of Eq. (7), and Eq. (2) into the right side after a slight modification. (Note that in the case of $i \to i$ scattering, the scattering angle is 0, i.e., $\cos\theta = 1$, so $P_j(1) = 1$.)

$$\begin{aligned}
\text{Im }\mathcal{M}(i \to i) = 2p\sqrt{s}\sigma_{\text{tot}} \geq 2p\sqrt{s}\sigma_{2\to 2} \to 16\pi\sum_{j=0}^{\infty}(2j+1)\text{Im }a_j &\geq 32\pi\frac{p}{\sqrt{s}}\sum_{j=0}^{\infty}(2j+1)|a_j|^2 \\
\sum_{j=0}^{\infty}(2j+1)\left[\text{Im }a_j - 2\frac{p}{\sqrt{s}}|a_j|^2\right] &\geq 0
\end{aligned} \tag{8}$$

Now, although it is not strictly rigorous, we make two assumptions to obtain a practical equation. The first assumption is that $\text{Im }a_j - 2\frac{p}{\sqrt{s}}|a_j|^2 \geq 0$ holds for each j value. The second assumption is to assume

[3] Refer to $\int d\cos\theta P_j(\cos\theta)P_k(\cos\theta) = \frac{2\delta_{jk}}{2j+1}$.

[4] Refer to the fact that $d\sigma = \frac{|\mathcal{M}|^2}{4p\sqrt{s}}d\Pi$ in the CM frame.

ultra-high-energy scattering. In this case, $2\frac{p}{\sqrt{s}} \to 1$. The second assumption is appropriate because in any UV-complete theory, we can increase the scattering energy infinitely, and we will handle such limits later.
Now, for each partial wave amplitude prefactor a_j in ultra-high-energy scattering, the following holds:

$$\text{Im}\ a_j \geq |a_j|^2 \tag{9}$$

This inequality forces the following:[5]

$$|a_j| \leq 1 \tag{10}$$

This equation is what we will importantly use later.
Let's take ϕ^4 theory: $\mathcal{L} = -\phi(\Box + m^2)\phi - \frac{\lambda}{4!}\phi^4$ as an example. What is the maximum value of λ? Of course, in any interacting theory, the interaction strength (here λ) must be small enough to justify doing perturbative expansion, but setting that aside, we can see λ are constrained by unitarity as follows:

$$\mathcal{M} = \lambda + \mathcal{O}(\lambda^2) = 16\pi \sum_{j=0}^{\infty} a_j(2j+1)P_j(\cos\theta) \tag{11}$$

In the case of ϕ^4 theory, the amplitude does not have angular dependency, so $P_j = P_0\delta_{0j}$.

$$\lambda + \mathcal{O}(\lambda^2) = 16\pi a_0 \leq 16\pi \tag{12}$$

Therefore, we can see that $\lambda \leq 16\pi$ must hold due to unitarity.[6]

2 Higgs mechanism

We revisit how gauge bosons acquire mass through spontaneous symmetry breaking in this section. Especially, it is usually said that Goldstone bosons are <u>eaten</u> by the gauge bosons. We need to understand what this means for further discussion.
Let's consider the following Lagrangian. Here $F_{\mu\nu}$ is the gauge boson corresponding to U(1) symmetry, and φ is a complex scalar particle charged under this U(1).

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu\varphi|^2 - V(\varphi) \tag{13}$$

$$D_\mu = \partial_\mu + iqA_\mu\ , \quad V(\varphi) = -\mu^2|\varphi|^2 + \lambda|\varphi|^4 \tag{14}$$

Let's expand φ as follows:

$$\varphi = \frac{1}{\sqrt{2}}(v + \phi_1 + i\phi_2) \quad \left(v = \sqrt{\frac{-\mu^2}{\lambda}}\right) \tag{15}$$

[5]For $a_j = R + iI$,

$$I \geq R^2 + I^2 \geq I^2 \to |I| \geq |I^2| \to 1 \geq |I|$$

[6]Comments on whether we can really ignore the $\mathcal{O}(\lambda^2)$ term (especially when λ is greater than 1), and the possibility of a negative $\mathcal{O}(\lambda^2)$ term which relaxes the perturbative bound. Within $2 \to 2$ process, the $\mathcal{O}(\lambda^2)$ correction is made by loops, and loop-involved amplitudes are usually suppressed by a $(16\pi^2)^{-1}$ factor which outweighs 16π, and at least in ϕ^4 theory, the $\mathcal{O}(\lambda^2)$ correction is positive.

It is hard to see the Lagrangian can be expanded as follows.[7] (Any textbook saying 'it is easy to see ⋯' is lying.)

$$\mathcal{L} = \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(qv)^2 A_\mu^2\right) + \left(\frac{1}{2}(\partial\phi_1)^2 - \frac{2\lambda v^2}{2}\phi^2\right) + \left(\frac{1}{2}(\partial\phi_2)^2\right) + qvA_\mu\partial^\mu\phi_2$$
$$+ \text{(interaction terms)}$$

As you can see, the gauge field A_μ has acquired a mass of qv due to the symmetry breaking of the complex scalar. Also, a massless Goldstone boson ϕ_2 has appeared. The problem in the Lagrangian is the mysterious quadratic term $qvA_\mu\partial^\mu\phi^2$: this looks like mass mixing between the two fields A_μ and $\partial\phi_2$, and thus needs to be removed.

This removal is achieved through gauge fixing. Specifically, let's add the following gauge fixing term:

$$\mathcal{L}_{\text{gf}} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi qv\phi_2)^2 = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 + qv\phi_2\partial_\mu A^\mu - \frac{1}{2}\xi(qv)^2\phi_2^2 \tag{16}$$

$$\mathcal{L} + \mathcal{L}_{\text{gf}} = \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(qv)^2 A_\mu^2\right) + \left(\frac{1}{2}(\partial\phi_1)^2 - \frac{2\lambda v^2}{2}\phi^2\right) + \left(\frac{1}{2}(\partial\phi_2)^2 - \frac{1}{2}\xi(qv)^2\phi_2^2\right)$$
$$- \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \cancel{qv\partial^\mu(A_\mu\phi_2)} + \text{(interaction terms)} \tag{17}$$

By adding the gauge fixing term, the problematic quadratic term becomes a total derivative and disappears, and the Goldstone boson ϕ_2 acquires a gauge dependent mass $\xi(qv)^2$.

So what does 'the Goldstone boson is eaten by the gauge boson' mean? It means we take the unitary gauge: $\xi \to \infty$. We indeed can do this as any physical observables are gauge independent. Within unitary gauge, ϕ_2 has an infinite mass and therefore cannot have any influence on the dynamics of A_μ and ϕ_1. In other words, we can handle the physics while ignoring it. Of course, we could choose another gauge where ξ is not infinite, in which case we would have to consider ϕ_2 in the gauge field dynamics. Do we need to choose another gauge? In fact, this is the core of the Goldstone equivalence theorem.

Some of you may have had experience expanding as $\varphi = \frac{1}{2}(v + \phi_1)$. This is an expansion that ignores ϕ_2, and naturally, it yields the same results as the limit where ϕ_2 is absent: the unitary gauge.

The above discussion is identical for non-abelian theories. In the Standard Model, three Goldstone bosons appear, and by choosing the unitary gauge where $\xi \to \infty$, they disappear while acquiring infinite mass. However, there is one major difference due to being non-abelian, which is that all particles charged under non-abelian (such as SU(2)) fields have the same charge, i.e., the same interaction strength.

Why is this point important? If you deal with dark matter, you might have heard of things like dark photon models. If the mass of the dark photon originates from a new dark Higgs, the dark Higgs could also act as a portal between DM and SM. However, in many dark photon analyses, the influence of the dark Higgs is not well considered. This can be justified by making the mass of the dark Higgs ϕ_1 very high. That is, the VEV v must be high enough, and the charge q of the dark Higgs must be assumed to be very low (compared to the charge of dark matter under the dark photon). However, in the case of non-abelian theory, if both dark matter and dark Higgs are charged under the dark photon, they must have the same coupling, making the above evasion difficult.

[7] $\text{(interaction terms)} = qA_\mu(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1) + \frac{q^2}{2}A_\mu^2(2v\phi_1 + \phi_1^2 + \phi_2^2) - \lambda v\phi_1^3 - \lambda v\phi_1\phi_2^2 - \frac{\lambda}{4}(\phi_1^2 + \phi^2)^2$

3 $WWWW$ Scattering

SM Lagrangian for EW and Higgs Boson

Now let's deal with the main point of this lecture: the Standard Model. The Electroweak Lagrangian without quarks is as follows:

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D_\mu H)^\dagger (D^\mu H) - \mu^2 H^\dagger H - \lambda (H^\dagger H)^2 \tag{18}$$

$$D_\mu H \equiv \left(\partial_\mu - \frac{i}{2} g W^i_\mu \tau^i - \frac{i}{2} g' B_\mu\right) H$$

For now, let's proceed without considering Goldstone bosons. That is, we substitute $H = \frac{1}{\sqrt{2}}\binom{0}{v+h}$.
The interaction Lagrangian related to the $W^+W^- \to W^+W^-$ process is as follows. [1] [8]

$$\begin{aligned}\mathcal{L} \supset\ & ig\cos\theta_w[(\partial_\mu Z_\nu - \partial_\nu Z_\mu)W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)Z_\mu W^-_\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)Z_\mu W^+_\nu] \\ & + ie[(\partial_\mu A_\nu - \partial_\nu A_\mu)W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)A_\mu W^-_\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)A_\mu W^+_\nu] \\ & + \frac{g^2}{2}(W^+_\mu W^+_\mu W^-_\nu W^-_\nu - W^+_\mu W^-_\mu W^-_\nu W^+_\nu) \\ & + g m_W h W^+_\mu W^-_\mu \end{aligned} \tag{19}$$

Where

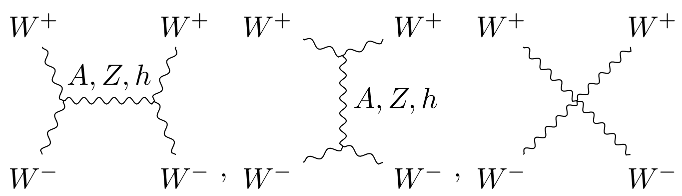
$$\tan\theta_w = \frac{g'}{g} \to \cos\theta_w = \frac{g}{\sqrt{g^2+g'^2}},\ \sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}} \tag{20}$$

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix}\begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix},\ W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp i W^2_\mu) \tag{21}$$

$$m_W = \frac{gv}{2},\ m_Z = \frac{gv}{2\cos\theta_w} = \frac{m_W}{\cos\theta_w} = \frac{\sqrt{g^2+g'^2}v}{2} \tag{22}$$

$$e = g\sin\theta_w = g'\cos\theta_w \tag{23}$$

At the tree-level, the following 7 diagrams contribute to the $W^+W^- \to W^+W^-$ process.



Polarization of External Gauge Bosons

Now let's evaluate the $WWWW$ scattering amplitude! We must remember that external polarizations are involved in the scattering amplitude. To obtain those polarization vectors, we need to know a little bit about the scattering kinematics. Consider the COM frame on the xz plane. (See Figure 1.)
The 4-momentum of each gauge boson is as follows.

$$\begin{aligned} p_1^\mu &= (E,0,0,p), \quad p_2^\mu = (E,0,0,-p), \\ p_3^\mu &= (E, p\sin\theta, 0, p\cos\theta),\ p_4^\mu = (E, -p\sin\theta, 0, -p\cos\theta) \end{aligned} \tag{24}$$

[8] This book does not strictly consider the upper and lower positions of Lorentz indices, which is standard in tools like FeynCalc.

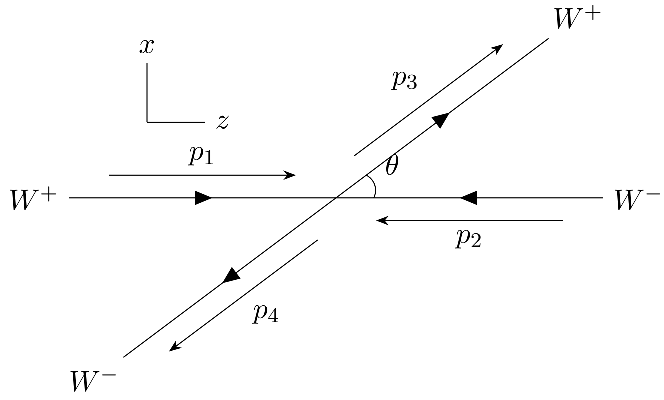


Figure 1: $WWWW$ Scattering

The polarization vectors corresponding to each 4-momentum are as follows. (Determined by the conditions $\epsilon(k)\cdot k = 0$ and $\epsilon_i(k)\cdot\epsilon_j(k) = -\delta_{ij}$.)

$$\epsilon_1^\mu(p_1) = (0,1,0,0),\quad \epsilon_2^\mu(p_1) = (0,0,1,0),\quad \epsilon_L^\mu(p_1) = \frac{1}{m_W}(p,0,0,E)$$
$$\epsilon_1^\mu(p_2) = (0,1,0,0),\quad \epsilon_2^\mu(p_2) = (0,0,1,0),\quad \epsilon_L^\mu(p_2) = \frac{1}{m_W}(p,0,0,-E)$$
$$\epsilon_1^\mu(p_3) = (0,\cos\theta,0,-\sin\theta),\ \epsilon_2^\mu(p_3) = (0,0,1,0),\ \epsilon_L^\mu(p_3) = \frac{1}{m_W}(p,E\sin\theta,0,E\cos\theta)$$
$$\epsilon_1^\mu(p_4) = (0,\cos\theta,0,-\sin\theta),\ \epsilon_2^\mu(p_4) = (0,0,1,0),\ \epsilon_L^\mu(p_4) = \frac{1}{m_W}(p,-E\sin\theta,0,-E\cos\theta)$$

Now, you can see that momentum and energy enter into the longitudinal polarization. According to our earlier unitarity discussion, the scattering amplitude must be below a certain value at any energy. For longitudinal polarization, however, it seems like the strength of the amplitude will increase as the energy gets higher. This is the point. We deal with the extremal case: The case where all external W bosons have longitudinal polarizations.

Since we will describe the amplitude in terms of the Lorentz invariant quantity, Mandelstam variables, the relationship between E, p, θ and s, t is as follows.

$$E^2 = \frac{s}{4},\ p^2 = E^2 - m_W^2 = \frac{s}{4} - m_W^2,\ \cos\theta = 1 + \frac{t}{2p^2} = 1 + \frac{2t}{s - 4m_W^2} \tag{25}$$

I use Mathematica for the substitution to Mandelstam variables.

Amplitude without Higgs - s-channel

It is time to evaluate the $W_L W_L \to W_L W_L$ amplitude. To emphasize the role of the Higgs, we will ignore the Higgs for now.

One thing to note before calculating: we use implicitly the Unitary gauge here. The Z propagator in this gauge is as follows.

$$i\Pi^{\mu\nu} = \frac{-i}{p^2 - m_Z^2}\left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{m_Z^2}\right) \tag{26}$$

The calculation process for the s-channel amplitude mediated by the photon and Z boson is as follows.

(To skip to the answer, refer to Eq. (32).) Here, we denote $\epsilon_L^\alpha(p_i) \to \epsilon_i^\alpha$, and $k = p_1 + p_2$.

$$
\begin{aligned}
i\mathcal{M}_{s,AZ} = \epsilon_1^\alpha \epsilon_2^\beta \epsilon_3^{*\mu} \epsilon_4^{*\nu} &\left[\eta^{\mu\nu}(p_3-p_4)^\lambda + \eta^{\nu\lambda}(k+p_4)^\mu - \eta^{\lambda\mu}(p_3+k)^\nu\right] \\
&\left[(ie)^2\frac{-ig_{\lambda\gamma}}{s} + (ig\cos\theta_W)^2\frac{-i}{s-m_Z^2}\left(g_{\lambda\gamma} - \frac{k_\lambda k_\gamma}{s}\right)\right] \\
&\left[\eta^{\alpha\beta}(p_1-p_2)^\gamma + \eta^{\beta\gamma}(k+p_2)^\alpha - \eta^{\gamma\alpha}(p_1+k)^\beta\right]
\end{aligned} \tag{27}
$$

Terms proportional to $k_\lambda k_\gamma$ are zero for the following reason. (Also refer to $k = p_3 + p_4$ and $\epsilon(q)\cdot q = 0$.)

$$
\begin{aligned}
(p_3-p_4)^\lambda k_\lambda = p_3^2 - p_4^2 = 0 \\
\epsilon_3^{*\mu}\epsilon_4^{*\nu}(\eta^{\nu\lambda}(k+p_4)^\mu - \eta^{\lambda\mu}(p_3+k)^\nu)k_\lambda = \epsilon_3^{*\mu}\epsilon_4^{*\nu}(k^\nu(k+p_4)^\mu - k^\mu(p_3+k)^\nu) \\
= (\epsilon_3^*\cdot(k+p_4))(\epsilon_4^*\cdot k) - (\epsilon_3^*\cdot k)(\epsilon_4^*\cdot(k+p_3)) = (\epsilon_3^*\cdot 2(p_4))(\epsilon_4^*\cdot p_3) - (\epsilon_3^*\cdot p_4)(\epsilon_4^*\cdot(2p_3)) = 0
\end{aligned}
$$

Eq. (27) is now simplified as follows. $\epsilon(q)\cdot q = 0$ is used again.

$$
\begin{aligned}
i\mathcal{M}_{s,AZ} = &\, i\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right] g_{\lambda\gamma}\epsilon_1^\alpha\epsilon_2^\beta\left[\eta^{\alpha\beta}(p_1-p_2)^\gamma + \eta^{\beta\gamma}(k+p_2)^\alpha - \eta^{\gamma\alpha}(p_1+k)^\beta\right] \\
&\epsilon_3^{*\mu}\epsilon_4^{*\nu}\left[\eta^{\mu\nu}(p_3-p_4)^\lambda + \eta^{\nu\lambda}(k+p_4)^\mu - \eta^{\lambda\mu}(p_3+k)^\nu\right] \\
= &\, i\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right] g_{\lambda\gamma}\epsilon_1^\alpha\epsilon_2^\beta\left[\eta^{\alpha\beta}(p_1-p_2)^\gamma + 2\eta^{\beta\gamma}p_2^\alpha - 2\eta^{\gamma\alpha}p_1^\beta\right] \\
&\epsilon_3^{*\mu}\epsilon_4^{*\nu}\left[\eta^{\mu\nu}(p_3-p_4)^\lambda + 2\eta^{\nu\lambda}p_4^\mu - 2\eta^{\lambda\mu}p_3^\nu\right] \\
= &\, i\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right] g_{\lambda\gamma}\left[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)^\gamma + 2\epsilon_2^\gamma(\epsilon_1\cdot p_2) - 2\epsilon_1^\gamma(\epsilon_2\cdot p_1)\right] \\
&\left[(\epsilon_3^*\cdot\epsilon_4^*)(p_3-p_4)^\lambda + 2\epsilon_4^{*\lambda}(\epsilon_3^*\cdot p_4) - 2\epsilon_3^{*\lambda}(\epsilon_4^*\cdot p_3)\right]
\end{aligned} \tag{28}
$$

Now, let's substitute Lorentz invariant products like $(\epsilon_1\cdot\epsilon_2)$ as follows.

$$
(\epsilon_1\cdot\epsilon_2) = (\epsilon_3^*\cdot\epsilon_4^*) = \frac{p^2+E^2}{m_W^2} \tag{29}
$$

$$
(\epsilon_1\cdot p_2) = (\epsilon_2\cdot p_1) = (\epsilon_3\cdot p_4) = (\epsilon_4\cdot p_3) = \frac{2pE}{m_W} \tag{30}
$$

Substituting the above into Eq. (28), we get

$$
\begin{aligned}
\mathcal{M}_{s,AZ} = &\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right] g_{\lambda\gamma}\left[\frac{p^2+E^2}{m_W^2}(p_1-p_2)^\gamma + \frac{4pE}{m_W}(\epsilon_2^\gamma - \epsilon_1^\gamma)\right] \\
&\left[\frac{p^2+E^2}{m_W^2}(p_3-p_4)^\lambda + \frac{4pE}{m_W}\left(\epsilon_4^{*\lambda} - \epsilon_3^{*\lambda}\right)\right] \\
= &\frac{1}{m_W^4}\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right]\left[(p^2+E^2)(p_1-p_2)^\lambda + 4m_W pE\left(\epsilon_2^\lambda - \epsilon_1^\lambda\right)\right] \\
&\left[(p^2+E^2)(p_3-p_4)^\lambda + 4m_W pE\left(\epsilon_4^{*\lambda} - \epsilon_3^{*\lambda}\right)\right]
\end{aligned} \tag{31}
$$

Now we use the following equations.

$$
\begin{aligned}
(p_1-p_2)\cdot(p_3-p_4) = -4p^2\cos\theta \\
\left(\epsilon_2^\lambda - \epsilon_1^\lambda\right)\cdot(p_3-p_4) = \left(\epsilon_4^{*\lambda} - \epsilon_3^{*\lambda}\right)\cdot(p_1-p_2) = \frac{4Ep}{m_W}\cos\theta \\
\left(\epsilon_2^\lambda - \epsilon_1^\lambda\right)\cdot\left(\epsilon_4^{*\lambda} - \epsilon_3^{*\lambda}\right) = -\frac{4E^2}{m_W^2}\cos\theta
\end{aligned}
$$

Substituting into Eq. (31), we obtain the following.

$$\mathcal{M}_{s,AZ} = \frac{1}{m_W^4}\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right]$$
$$\times\left[(p^2+E^2)^2(-4p^2\cos\theta) + 4m_W pE(p^2+E^2)\frac{8Ep}{m_W}\cos\theta + 16m_W^2p^2E^2\left(-\frac{4E^2}{m_W^2}\cos\theta\right)\right]$$
$$= \frac{1}{m_W^4}\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right]\left[-4p^2(p^2+E^2)^2 + 32E^2p^2(p^2+E^2) - 64p^2E^4\right]\cos\theta$$
$$= -\frac{4}{m_W^4}\left[\frac{e^2}{s} + \frac{g^2\cos^2\theta_W}{s-m_Z^2}\right]p^2(p^2-3E^2)^2\cos\theta$$
$$\mathcal{M}_{s,AZ} = -\frac{g^2}{4m_W^4}\left[\frac{\sin^2\theta_W}{s} + \frac{\cos^2\theta_W}{s-m_Z^2}\right](s+2m_W^2)^2(s+2t-4m_W^2) \tag{32}$$

High energy limit of amplitude - part 1

Before calculating the next amplitude, let's first look at the ultra-high-energy limit of Eq. (32). In the limit where $s \gg m_Z^2$, $\frac{1}{s-m_Z^2} = \frac{1}{s}\left(1 + \frac{m_Z^2}{s} + \frac{m_Z^4}{s^2} + \cdots\right)$. Therefore, $\mathcal{M}_{s,AZ}$ can be expanded as a polynomial in s and t, which is as follows. (In particular, we focus on the terms where $n+m \geq 0$ for $s^n t^m$.)

$$\begin{aligned}
\mathcal{M}_{s,AZ} &\approx -\frac{g^2}{4m_W^4}\left[\frac{\sin^2\theta_W}{s} + \frac{\cos^2\theta_W}{s} + \frac{\cos^2\theta_W m_Z^2}{s^2} + \frac{\cos^2\theta_W m_Z^4}{s^3}\right] \\
&\quad\times(s^3 + 2s^2t + 8m_W^2st - 12m_W^4s + 8m_W^4t - 16m_W^6) \\
&\approx -\frac{g^2}{4m_W^4}\left[1 + \frac{m_W^2}{s} + \frac{m_W^2m_Z^2}{s^2}\right]\left(s^2 + 2st + 8m_W^2t - 12m_W^4 + 8m_W^4\frac{t}{s}\right) \\
&\approx -\frac{g^2}{4m_W^4}\left(s^2 + m_W^2s + m_W^2m_Z^2 + 2st + 2m_W^2t + 2m_W^2m_Z^2\frac{t}{s}\right. \\
&\quad\left. + 8m_W^2t + 8m_W^4\frac{t}{s} - 12m_W^4 + 8m_W^4\frac{t}{s}\right)
\end{aligned} \tag{33}$$

$$\mathcal{M}_{s,AZ} \approx -\frac{g^2}{4m_W^4}\left(s^2 + 2st + m_W^2(s+10t) + m_W^2(m_Z^2 - 12m_W^2) + 2m_W^2(m_Z^2 + 8m_W^2)\frac{t}{s}\right) \tag{34}$$

The leading order is $\mathcal{O}(s^2)$. Concerning unitarity, this is dangerous as the amplitude must remain below a certain value regardless of the energy.

t-channel Amplitude

The calculation process for the t-channel amplitude mediated by the photon and Z boson is as follows. (To skip to the answer, see Eq. (48).)

$$\begin{aligned}
i\mathcal{M}_{t,AZ} = \epsilon_1^\alpha\epsilon_2^\beta\epsilon_3^{*\mu}\epsilon_4^{*\nu}&\left[\eta^{\alpha\mu}(p_1+p_3)^\gamma + \eta^{\mu\gamma}(k-p_3)^\alpha - \eta^{\gamma\alpha}(k+p_1)^\mu\right] \\
&\left[(ie)^2\frac{-ig_{\lambda\gamma}}{t} + (ig\cos\theta_W)^2\frac{-i}{t-m_Z^2}\left(g_{\lambda\gamma} - \frac{k_\lambda k_\gamma}{s}\right)\right] \\
&\left[-\eta^{\beta\nu}(p_2+p_4)^\lambda + \eta^{\nu\lambda}(k+p_4)^\beta + \eta^{\lambda\beta}(p_2-k)^\nu\right]
\end{aligned} \tag{35}$$

Again, terms proportional to $k_\lambda k_\gamma$ are zero.

$$\begin{aligned}
&\epsilon_1^\alpha\epsilon_3^{*\mu}\left[\eta^{\alpha\mu}(p_1+p_3)^\gamma + \eta^{\mu\gamma}(k-p_3)^\alpha - \eta^{\gamma\alpha}(k+p_1)^\mu\right]k_\gamma \\
=&(\epsilon_1\cdot\epsilon_3^*)(p_1+p_3)\cdot(p_1-p_3) + (\epsilon_3^*\cdot k)\,\epsilon_1\cdot(k-p_3) - (\epsilon_1\cdot k)\,\epsilon_3^*\cdot(k+p_1) \\
=&0 + (\epsilon_3^*\cdot p_1)\,\epsilon_1\cdot(-2p_3) - (\epsilon_1\cdot(-p_3))\,\epsilon_3^*\cdot(2p_1) = 0
\end{aligned}$$

$$\begin{aligned}
i\mathcal{M}_{t,AZ} = {} & \epsilon_1^\alpha \epsilon_3^{*\mu} \left[\eta^{\alpha\mu}(p_1+p_3)^\gamma + \eta^{\mu\gamma}(k-p_3)^\alpha - \eta^{\gamma\alpha}(k+p_1)^\mu\right] \\
& \left[(ie)^2 \frac{-i}{t} + (ig\cos\theta_W)^2 \frac{-i}{t-m_Z^2}\right] g_{\lambda\gamma} \\
& \epsilon_2^\beta \epsilon_4^{*\nu} \left[-\eta^{\beta\nu}(p_2+p_4)^\lambda + \eta^{\nu\lambda}(k+p_4)^\beta + \eta^{\lambda\beta}(p_2-k)^\nu\right] \qquad (36) \\
= {} & i\epsilon_1^\alpha \epsilon_3^{*\mu} \left[\eta^{\alpha\mu}(p_1+p_3)^\gamma + \eta^{\mu\gamma}(k-p_3)^\alpha - \eta^{\gamma\alpha}(k+p_1)^\mu\right] \left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] g_{\lambda\gamma} \\
& \epsilon_2^\beta \epsilon_4^{*\nu} \left[-\eta^{\beta\nu}(p_2+p_4)^\lambda + \eta^{\nu\lambda}(k+p_4)^\beta + \eta^{\lambda\beta}(p_2-k)^\nu\right] \qquad (37)
\end{aligned}$$

$$\begin{aligned}
\mathcal{M}_{t,AZ} = {} & \left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \epsilon_1^\alpha \epsilon_3^{*\mu} \left[\eta^{\alpha\mu}(p_1+p_3)^\gamma - 2\eta^{\mu\gamma}p_3^\alpha - 2\eta^{\gamma\alpha}p_1^\mu\right] g_{\lambda\gamma} \\
& \epsilon_2^\beta \epsilon_4^{*\nu} \left[-\eta^{\beta\nu}(p_2+p_4)^\lambda + 2\eta^{\nu\lambda}p_4^\beta + 2\eta^{\lambda\beta}p_2^\nu\right] \qquad (38) \\
= {} & \left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \left[(\epsilon_1\cdot\epsilon_3^*)(p_1+p_3)^\gamma - 2\epsilon_3^{*\gamma}(\epsilon_1\cdot p_3) - 2\epsilon_1^\gamma(\epsilon_3^*\cdot p_1)\right] g_{\lambda\gamma} \\
& \left[-(\epsilon_2\cdot\epsilon_4^*)(p_2+p_4)^\lambda + 2\epsilon_4^{*\lambda}(\epsilon_2\cdot p_4) + 2\epsilon_2^\lambda(\epsilon_4^*\cdot p_2)\right] \qquad (39) \\
= {} & -\left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \left[(\epsilon_1\cdot\epsilon_3^*)(p_1+p_3)^\lambda - 2\epsilon_3^{*\lambda}(\epsilon_1\cdot p_3) - 2\epsilon_1^\lambda(\epsilon_3^*\cdot p_1)\right] \\
& \left[(\epsilon_2\cdot\epsilon_4^*)(p_2+p_4)^\lambda - 2\epsilon_4^{*\lambda}(\epsilon_2\cdot p_4) - 2\epsilon_2^\lambda(\epsilon_4^*\cdot p_2)\right] \qquad (40)
\end{aligned}$$

Substitute other Lorentz invariant contractions.

$$(\epsilon_1\cdot\epsilon_3^*) = (\epsilon_2\cdot\epsilon_4^*) = \frac{p^2 - E^2\cos\theta}{m_W^2} \qquad (41)$$

$$(\epsilon_1\cdot p_3) = (\epsilon_3^*\cdot p_1) = (\epsilon_2\cdot p_4) = (\epsilon_4^*\cdot p_2) = \frac{Ep(1-\cos\theta)}{m_W} \qquad (42)$$

$$\begin{aligned}
\mathcal{M}_{t,AZ} = {} & -\left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \left[\frac{p^2-E^2\cos\theta}{m_W^2}(p_1+p_3)^\lambda - \frac{2Ep(1-\cos\theta)}{m_W}(\epsilon_3^{*\lambda}+\epsilon_1^\lambda)\right] \\
& \left[\frac{p^2-E^2\cos\theta}{m_W^2}(p_2+p_4)^\lambda - \frac{2Ep(1-\cos\theta)}{m_W}(\epsilon_4^{*\lambda}+\epsilon_2^\lambda)\right] \\
= {} & -\frac{1}{m_W^4}\left[\frac{e^2}{t} + \frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \left[(p^2-E^2\cos\theta)(p_1+p_3)^\lambda - 2m_W Ep(1-\cos\theta)(\epsilon_3^{*\lambda}+\epsilon_1^\lambda)\right] \\
& \left[(p^2-E^2\cos\theta)(p_2+p_4)^\lambda - 2m_W Ep(1-\cos\theta)(\epsilon_4^{*\lambda}+\epsilon_2^\lambda)\right] \qquad (43)
\end{aligned}$$

Substitute other Lorentz invariant contractions.

$$(p_1+p_3)\cdot(p_2+p_4) = 4E^2 + 2p^2(1+\cos\theta) \qquad (44)$$

$$(p_1+p_3)\cdot(\epsilon_4^*+\epsilon_2) = (p_2+p_4)\cdot(\epsilon_3^*+\epsilon_1) = \frac{2Ep}{m_W}(3+\cos\theta) \qquad (45)$$

$$(\epsilon_3^*+\epsilon_1)\cdot(\epsilon_4^*+\epsilon_2) = \frac{1}{m_W^2}(4p^2 + 2E^2(1+\cos\theta)) \qquad (46)$$

$$
\begin{aligned}
\mathcal{M}_{t,AZ} =& -\frac{1}{m_W^4}\left[\frac{e^2}{t}+\frac{g^2\cos^2\theta_W}{t-m_Z^2}\right] \\
&\bigg[(p^2-E^2\cos\theta)^2(4E^2+2p^2(1+\cos\theta)) - 4m_W Ep(1-\cos\theta)(p^2-E^2\cos\theta)\frac{2Ep}{m_W}(3+\cos\theta) \\
&+ 4m_W^2E^2p^2(1-\cos\theta)^2\frac{1}{m_W^2}(4p^2+2E^2(1+\cos\theta))\bigg] \\
=& -\frac{1}{m_W^4}\left[\frac{e^2}{t}+\frac{g^2\cos^2\theta_W}{t-m_Z^2}\right]\bigg[(p^2-E^2\cos\theta)^2(4E^2+2p^2(1+\cos\theta)) \\
&- 8E^2p^2(1-\cos\theta)(p^2-E^2\cos\theta)(3+\cos\theta) + 4E^2p^2(1-\cos\theta)^2(4p^2+2E^2(1+\cos\theta))\bigg]
\end{aligned} \tag{47}
$$

$$
\begin{aligned}
=& -\frac{1}{4m_W^4(s-4m_W^2)^2}\left[\frac{e^2}{t}+\frac{g^2\cos^2\theta_W}{t-m_Z^2}\right](s^2t^2(2s+t) - 8m_W^2s^2t(s+3t) \\
&+ 4m_W^4s(2s^2+21st+20t^2) - 16m_W^6s(5s+14t) + 64m_W^8(4s+t) - 256m_W^{10})
\end{aligned} \tag{48}
$$

High energy limit of amplitude - part 2

Let's consider the ultra-high-energy limit again. Note that $t = p^2\cos\theta$. So the following expansion is justified as $|t|$ will be sufficiently greater than m_W^2 (in the ultra-high-energy limit).

$$
\frac{1}{t-m_Z^2} \approx \frac{1}{t} + \frac{m_Z^2}{t^2} + \frac{m_Z^4}{t^3} \tag{49}
$$

$$
\frac{1}{(s-4m_W^2)^2} \approx \frac{1}{s^2}\left(1+\frac{4m_W^2}{s}+\frac{16m_W^4}{s^2}+\cdots\right)^2 \approx \frac{1}{s^2}\left(1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right) \tag{50}
$$

$$
\begin{aligned}
&\mathcal{M}_{t,AZ} \\
\approx& -\frac{1}{4m_W^4}\frac{1}{s^2}\left[1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right]\left[\frac{g^2\sin^2\theta_W}{t}+\frac{g^2\cos^2\theta_W}{t}+\frac{g^2\cos^2\theta_W m_Z^2}{t^2}+\frac{g^2\cos^2\theta_W m_Z^4}{t^3}\right] \\
&\big(s^2t^2(2s+t) - 8m_W^2s^2t(s+3t) + 4m_W^4s(2s^2+21st+20t^2) - 16m_W^6s(5s+14t) + \mathcal{O}(s,t)\big) \\
=& -\frac{g^2}{4m_W^4}\frac{1}{s^2t}\left[1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right]\left[1+\frac{m_W^2}{t}+\frac{m_W^2m_Z^2}{t^2}\right] \\
&\big(s^2t^2(2s+t) - 8m_W^2s^2t(s+3t) + 4m_W^4s(2s^2+21st+20t^2) + \mathcal{O}(s^2,st,t^2)\big)
\end{aligned} \tag{51}
$$

As with the previous s-channel results, terms smaller than $\frac{\mathcal{O}(s^2,st,t^2)}{s^2 t}$ are ignored.

$$
\begin{aligned}
-\frac{4m_W^4}{g^2}\mathcal{M}_{t,AZ} &\approx \left[1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right]\left[1+\frac{m_W^2}{t}+\frac{m_W^2 m_Z^2}{t^2}\right] \\
&\quad \left(t(2s+t)-8m_W^2(s+3t)+4m_W^4\left(2\frac{s}{t}+21+20\frac{t}{s}\right)\right) \\
&\approx \left[1+\frac{8m_W^2}{s}+\frac{m_W^2}{t}+\frac{48m_W^4}{s^2}+\frac{8m_W^4}{st}+\frac{m_W^2 m_Z^2}{t^2}\right] \\
&\quad \left(t(2s+t)-8m_W^2(s+3t)+4m_W^4\left(2\frac{s}{t}+21+20\frac{t}{s}\right)\right) \\
&\approx \Bigg[t(2s+t)-8m_W^2(s+3t)+4m_W^4\left(2\frac{s}{t}+21+20\frac{t}{s}\right) \\
&\quad +m_W^2\left(\frac{8}{s}+\frac{1}{t}\right)\left(t(2s+t)-8m_W^2(s+3t)\right)+m_W^4\left(\frac{48}{s^2}+\frac{8}{st}+\frac{m_Z^2}{m_W^2}\frac{1}{t^2}\right)\left(t(2s+t)\right)\Bigg] \\
&\approx t^2+2st+m_W^2\left(-6s-7t+8\frac{t^2}{s}\right)+m_W^4\left(12-\frac{8t}{s}+\frac{48t^2}{s^2}+\frac{m_Z^2}{m_W^2}+2\frac{m_Z^2}{m_W^2}\frac{s}{t}\right) \qquad (52)
\end{aligned}
$$

We see the amplitude is divergent with leading order of t^2 in the ultra-high-energy limit. This indeed is dangerous because of unitarity.

4-point Amplitude

Now let's calculate the 4-point amplitude. (To skip to the answer, refer to Eq. (58).)

$$
i\mathcal{M}_4 = ig^2\epsilon_1^\alpha\epsilon_2^\beta\epsilon_3^{*\mu}\epsilon_4^{*\nu}\left[2g_{\mu\beta}g_{\nu\alpha}-g_{\mu\alpha}g_{\beta\nu}-g_{\mu\nu}g_{\alpha\beta}\right] \qquad (53)
$$

$$
\mathcal{M}_4 = g^2\left[2(\epsilon_1\cdot\epsilon_4^*)(\epsilon_2\cdot\epsilon_3^*)-(\epsilon_1\cdot\epsilon_3^*)(\epsilon_2\cdot\epsilon_4^*)-(\epsilon_1\cdot\epsilon_2)(\epsilon_3^*\cdot\epsilon_4^*)\right] \qquad (54)
$$

Lorentz invariant contractions are as follows.

$$
(\epsilon_1\cdot\epsilon_2) = (\epsilon_3^*\cdot\epsilon_4^*) = \frac{p^2+E^2}{m_W^2} \qquad (55)
$$

$$
(\epsilon_1\cdot\epsilon_3^*) = (\epsilon_2\cdot\epsilon_4^*) = \frac{p^2-E^2\cos\theta}{m_W^2} \qquad (56)
$$

$$
(\epsilon_1\cdot\epsilon_4^*) = (\epsilon_2\cdot\epsilon_3^*) = \frac{p^2+E^2\cos\theta}{m_W^2} \qquad (57)
$$

We get 4-point amplitude as follows.

$$
\begin{aligned}
\mathcal{M}_4 &= \frac{g^2}{m_W^4}\left[2(p^2+E^2\cos\theta)^2-(p^2-E^2\cos\theta)^2-(p^2+E^2)^2\right] \\
&= \frac{g^2}{4m_W^4}\frac{s}{(s-4m_W^2)^2}\left(s^3+4s^2t+st^2-4m_W^2(3s^2+7st)+48m_W^4(s+t)-64m_W^6\right) \qquad (58)
\end{aligned}
$$

High energy limit of amplitude - part 3

As before, let's consider the high-energy limit of the amplitude Eq. (58).

$$\begin{aligned}\mathcal{M}_4 \approx& \frac{g^2}{4m_W^4}\frac{s}{s^2}\left(1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right)\\&\left(s^3+4s^2t+st^2-4m_W^2(3s^2+7st)+48m_W^4(s+t)-64m_W^6\right) \quad (59)\\ \approx&\frac{g^2}{4m_W^4}\left(1+\frac{8m_W^2}{s}+\frac{48m_W^4}{s^2}\right)\\&\left(s^2+4st+t^2-4m_W^2(3s+7t)+48m_W^4\left(1+\frac{t}{s}\right)+\mathcal{O}\left(\frac{1}{s}\right)\right) \quad (60)\\ \frac{4m_W^4}{g^2}\mathcal{M}_4\approx& s^2+4st+t^2+m_W^2\left(-12s-28t+8s+32t+8\frac{t^2}{s}\right)\\&+m_W^4\left(48+192\frac{t}{s}+48\frac{t^2}{s^2}-96-224\frac{t}{s}+48+48\frac{t}{s}\right)+\mathcal{O}\left(\frac{1}{s},\frac{1}{t}\right) \quad (61)\\ \approx& s^2+4st+t^2+m_W^2\left(-4s+4t+8\frac{t^2}{s}\right)+m_W^4\left(16\frac{t}{s}+48\frac{t^2}{s^2}\right) \quad (62)\end{aligned}$$

High Energy Limit Amplitude without Higgs

At this intermediate stage, let's calculate the sum of the three amplitudes in the high-energy limit: Eq. (34), (52), and (62), which ignores the Higgs.[9]

$$\begin{aligned}&\frac{4m_W^4}{g^2}\mathcal{M}' \quad (63)\\ =&\frac{4m_W^4}{g^2}(\mathcal{M}_{s,AZ}+\mathcal{M}_{t,AZ}+\mathcal{M}_4)\\ \approx&-\left(s^2+2st+m_W^2(s+10t)+m_W^2(m_Z^2-12m_W^2)+2m_W^2(m_Z^2+8m_W^2)\frac{t}{s}\right)\\&-\left(t^2+2st+m_W^2\left(-6s-7t+8\frac{t^2}{s}\right)+m_W^4\left(12-\frac{8t}{s}+\frac{48t^2}{s^2}+\frac{m_Z^2}{m_W^2}+2\frac{m_Z^2}{m_W^2}\frac{s}{t}\right)\right)\\&+s^2+4st+t^2+m_W^2\left(-4s+4t+8\frac{t^2}{s}\right)+m_W^4\left(16\frac{t}{s}+48\frac{t^2}{s^2}\right)\\ =&\left[-(s^2+2st)-(t^2+2st)+s^2+4st+t^2\right]\\&+m_W^2\left[-(s+10t)+\left(6s+7t-8\frac{t^2}{s}\right)+\left(-4s+4t+8\frac{t^2}{s}\right)\right]\\&+m_W^4\left[-\left(\frac{m_Z^2}{m_W^2}-12\right)-2\left(\frac{m_Z^2}{m_W^2}+8\right)\frac{t}{s}-\left(12-\frac{8t}{s}+\frac{48t^2}{s^2}+\frac{m_Z^2}{m_W^2}+2\frac{m_Z^2}{m_W^2}\frac{s}{t}\right)+\left(16\frac{t}{s}+48\frac{t^2}{s^2}\right)\right]\end{aligned}$$

$$\frac{4m_W^4}{g^2}\mathcal{M}'=m_W^2(s+t)+m_W^4\left(-2\frac{m_Z^2}{m_W^2}-2\frac{m_Z^2}{m_W^2}\frac{s}{t}+\frac{t}{s}\left(8-2\frac{m_Z^2}{m_W^2}\right)\right)+\mathcal{O}\left(\frac{1}{s},\frac{1}{t}\right) \quad (64)$$

Fortunately(?), the dangerous $\mathcal{M}\sim\mathcal{O}(s^2)$ behavior we saw at the beginning has disappeared. The sum of the s, t, and 4-point amplitudes exactly canceled them out. However, the amplitude still exhibits $\mathcal{M}\sim\mathcal{O}(s)+\mathcal{O}(t)$ behavior, and thus still violates unitarity.

[9] Physically, this corresponds to the limit where $m_H \gg s \gg m_{W,Z}$.

Higgs Amplitude

Now it is time to consider the final piece, the contribution to the $WWWW$ amplitude from the Higgs. The s- and t-channel amplitudes mediated by the Higgs are as follows:

$$\begin{aligned} i\mathcal{M}_h &= (igm_W)^2 \epsilon_1^\alpha \epsilon_2^\beta \epsilon_3^{*\mu} \epsilon_4^{*\nu} \left[g_{\alpha\beta} g_{\mu\nu} \frac{i}{s-m_h^2} + g_{\alpha\mu} g_{\beta\nu} \frac{i}{t-m_h^2} \right] \\ &= -ig^2 m_W^2 \left[\frac{(\epsilon_1\cdot\epsilon_2)(\epsilon_3^*\cdot\epsilon_4^*)}{s-m_h^2} + \frac{(\epsilon_1\cdot\epsilon_3^*)(\epsilon_2\cdot\epsilon_4^*)}{t-m_h^2} \right] \\ &= -ig^2 m_W^2 \left[\frac{1}{s-m_h^2} \frac{(p^2+E^2)^2}{m_W^4} + \frac{1}{t-m_h^2} \frac{(p^2-E^2\cos\theta)^2}{m_W^4} \right] \\ &= -\frac{ig^2}{m_W^2} \left[\frac{1}{s-m_h^2} \frac{(s-2m_W^2)^2}{4} + \frac{1}{t-m_h^2} \frac{(st+2m_W^2 s - 8m_W^4)^2}{4(s-4m_W^2)^2} \right] \end{aligned} \tag{65}$$

$$\mathcal{M}_h = -\frac{g^2}{4m_W^2} \left[\frac{(s-2m_W^2)^2}{s-m_h^2} + \frac{(st+2m_W^2 s-8m_W^4)^2}{(t-m_h^2)(s-4m_W^2)^2} \right] \tag{66}$$

High energy limit of amplitude - part 4

The high-energy limit ($s \gg m_h^2$) of Eq. (66) is as follows. Again, we only expand up to the $s^0 t^0$ order.

$$\begin{aligned} &-\frac{4m_W^2}{g^2}\mathcal{M}_h \\ \approx& \left[\frac{1}{s}\left(1+\frac{m_h^2}{s}\right)(s-2m_W^2)^2 + \frac{1}{s^2 t}\left(1+\frac{m_h^2}{t}\right)\left(1+\frac{8m_W^2}{s}\right)(st+2m_W^2 s-8m_W^4)^2 \right] \\ \approx& s+t+2m_h^2+8m_W^2\frac{t}{s} \end{aligned} \tag{67}$$

Divergence Cancellation due to Higgs

Now, we can show that Eq. (67) precisely cancels the divergence in Eq. (64)!

$$\begin{aligned} \mathcal{M} &= \mathcal{M}' + \mathcal{M}_h \\ &= \frac{g^2}{4m_W^4}\left(m_W^2(s+t) + m_W^4\left(-2\frac{m_Z^2}{m_W^2} - 2\frac{m_Z^2}{m_W^2}\frac{s}{t} + \frac{t}{s}\left(8-2\frac{m_Z^2}{m_W^2}\right)\right)\right) \\ &- \frac{g^2}{4m_W^2}\left(s+t+2m_h^2+8m_W^2\frac{t}{s}\right) + \mathcal{O}\left(\frac{1}{s},\frac{1}{t}\right) \\ &= -\frac{g^2}{2m_W^2}m_h^2 + \frac{g^2}{4}\left(-2\frac{m_Z^2}{m_W^2} - 2\frac{m_Z^2}{m_W^2}\frac{s}{t} + \frac{t}{s}\left(8-2\frac{m_Z^2}{m_W^2}\right) - 8\frac{t}{s}\right) \\ &= -\frac{g^2}{2m_W^2}m_h^2 - \frac{g^2}{2}\frac{m_Z^2}{m_W^2}\left(1+\frac{s}{t}+\frac{t}{s}\right) \end{aligned} \tag{68}$$

As you can see, the Higgs causes the amplitude to behave as $\mathcal{O} \sim (s^n t^m)$ (where $n+m=0$). In other words, the amplitude no longer diverges even as the energy increases.

4 Lee-Quigg-Thacker Bound [3]

Now let's apply the unitarity bound Eq. (10) to Eq. (68).
First, substitute $t = \frac{1}{2}(s-4m_W^2)(\cos\theta - 1)$ and $\frac{m_Z^2}{m_W^2} = \sec^2\theta_w$ into Eq. (68).

$$\mathcal{M} \approx -\frac{g^2}{2m_W^2}m_h^2 - \frac{g^2}{2\cos^2\theta_w}\left(1 - \frac{2s}{(s-4m_W^2)(1-\cos\theta)} - \frac{(s-4m_W^2)(1-\cos\theta)}{2s}\right) \tag{69}$$

Here we will focus on the part where $l=0$, which is independent of $\cos\theta$. To do this, we use $(1-\cos\theta)^{-1} = 1+\cos\theta+\cdots$.

$$\mathcal{M} \approx -\frac{g^2}{2m_W^2}m_h^2 - \frac{g^2}{2\cos^2\theta_w}\left(1-\frac{2s}{(s-4m_W^2)}-\frac{(s-4m_W^2)}{2s}\right)+\mathcal{O}(\cos\theta)$$
$$\sim -\frac{g^2}{2m_W^2}m_h^2 - \frac{3}{2}\frac{g^2}{2\cos^2\theta_w} = 16\pi a_0 \tag{70}$$

(In this process, the $\mathcal{O}(m_W^2/s)$ correction was also ignored.) Now, since $a_0 \le 1$, we can see that the following inequality holds.

$$\frac{1}{32\pi}\frac{g^2}{m_W^2}m_h^2 < \frac{1}{16\pi}\left(\frac{g^2}{2m_W^2}m_h^2+\frac{3g^2}{4\cos^2\theta_w}\right)\le 1 \tag{71}$$

(The leftmost part is because $\frac{3g^2}{4\cos^2\theta_w}$ is positive.) Therefore, the following bound holds for the Higgs mass.

$$m_h < \sqrt{32\pi}\frac{m_W}{g} \tag{72}$$

Historically, when Lee, Quigg, and Thacker derived the above bound, neither the mass of the W boson nor the value of the electroweak coupling g was known. However, their combination g/m_W was very well known as the Fermi constant.[10]

$$G_F = \frac{1}{4\sqrt{2}}\frac{g^2}{m_W^2} \simeq 1.1166\times 10^{-5}\ \mathrm{GeV}^{-2} \tag{73}$$

$$\frac{1}{16\pi}\frac{4\sqrt{2}}{2}G_F m_h^2 = \frac{\sqrt{2}}{8\pi}G_F m_h^2 \le 1 \ \to m_h \le \sqrt{\frac{4\pi}{\sqrt{2}}}G_F^{-1/2} \simeq 1234.56\ \mathrm{GeV} \tag{74}$$

5 Goldstone Equivalence Theorem

The bound given by Lee, Quigg, and Thacker was actually calculated from a superposed state of the four particles W, Z, A, h as the initial state.

$$m_h^2 \le \frac{8\pi\sqrt{2}}{3G_F} \simeq 1008.0\mathrm{GeV} \tag{75}$$

This means they also calculated other scattering processes, such as $WZ \to WZ$.

This calculation is not for human. Because the kinematics become extremely complicated just by changing from $AA \to AA$ to $AB \to AB$. For instance, with $\epsilon_1\cdot\epsilon_3 = \frac{pp'-E^2\cos\theta}{m_A m_B}$, the moment there are two momentum values to handle, the work becomes extremely complex. Moreover, $AB \to AB$ is not all. One could also consider processes like $WW \to Z\gamma$. This means handling 3 momentum values.

Fortunately, there is a workaround called the Goldstone Equivalence theorem. Instead of proving the Goldstone Equivalence theorem here, I would like to explain it by giving direct examples.

Lagrangian in Feynman-'t Hooft Gauge

We used the unitary gauge ($\xi\to\infty$) in the previous section. This time, instead of the unitary gauge, let's use the Feynman-'t Hooft Gauge where $\xi = 1$. In this case, we must substitute $H = \frac{1}{\sqrt{2}}\binom{\sqrt{2}\phi^+}{v+h+i\phi^0}$. Also, each Goldstone boson has the same mass as the gauge fields W, Z (just as the mass of the Goldstone boson due to the gauge fixing term in the abelian case is $\sqrt{\xi}m_A$). (We will see this shortly.)

[10] This value was well measured through beta decay.

Choosing Feynman-'t Hooft Gauge does not change the form of interactions between gauge bosons.

$$\begin{aligned}
\mathcal{L} \supset\, & ig\cos\theta_w[(\partial_\mu Z_\nu - \partial_\nu Z_\mu)W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)Z_\mu W^-_\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)Z_\mu W^+_\nu] \\
& + ie[(\partial_\mu A_\nu - \partial_\nu A_\mu)W^+_\mu W^-_\nu - (\partial_\mu W^+_\nu - \partial_\nu W^+_\mu)A_\mu W^-_\nu + (\partial_\mu W^-_\nu - \partial_\nu W^-_\mu)A_\mu W^+_\nu] \\
& + \frac{g^2}{2}(W^+_\mu W^+_\mu W^-_\nu W^-_\nu - W^+_\mu W^-_\mu W^-_\nu W^+_\nu) + e^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^-_\nu W^+_\nu) \\
& + g^2\cos^2\theta_w(Z_\mu W^+_\mu Z_\nu W^-_\nu - Z_\mu Z_\mu W^-_\nu W^+_\nu) \\
& + eg\cos\theta_w(W^+_\mu W^-_\nu A_\mu Z_\nu + W^+_\nu W^-_\mu A_\mu Z_\nu - 2W^+_\mu W^-_\mu A_\nu Z_\nu)
\end{aligned} \tag{76}$$

The interactions with the Higgs boson, however, are changed drastically. Here $c(s)_w$ correspond to $\cos(\sin)\theta_w$, and A' and Z' denote $\sqrt{g^2+g'^2}A$ and $\sqrt{g^2+g'^2}Z$ respectively. To avoid double scripts, I have used ϕ_0 and ϕ^0 interchangeably.

$$\begin{aligned}
\mathcal{L}_H =\, & \left|\partial_\mu H - igW^a_\mu \tau^a H - \frac{ig'}{2}B_\mu H\right|^2 + \lambda v^2 H^\dagger H - \lambda (H^\dagger H)^2 \\
=\, & \left|\partial_\mu \begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(h+i\phi^0)\end{pmatrix} - \frac{i}{2}\begin{pmatrix} 2s_w c_w A'_\mu + (c_w^2 - s_w^2)Z'_\mu & \sqrt{2}gW^+_\mu \\ \sqrt{2}gW^-_\mu & -Z'_\mu \end{pmatrix}\begin{pmatrix} \phi^+ \\ \frac{1}{\sqrt{2}}(v+h+i\phi^0)\end{pmatrix}\right|^2 \\
& + \lambda v^2\left(\phi^+\phi^- + \frac{1}{2}(v+h)^2 + \frac{1}{2}\phi_0^2\right) - \lambda\left(\phi^+\phi^- + \frac{1}{2}(v+h)^2 + \frac{1}{2}\phi_0^2\right)^2 \\
=\, & \left|\partial_\mu\phi^+ - \frac{i}{2}\left((2s_w c_w A'_\mu + (c_w^2 - s_w^2)Z'_\mu)\phi^+ + gW^+_\mu(v+h+i\phi^0)\right)\right|^2 \\
& + \left|\frac{1}{\sqrt{2}}\partial_\mu(h+i\phi^0) - \frac{i}{2}\left((\sqrt{2}gW^-_\mu\phi^+ - \frac{1}{\sqrt{2}}Z'_\mu(v+h+i\phi^0))\right)\right|^2 \\
& + \frac{\lambda v^4}{4} - \frac{1}{2}2\lambda v^2\, h^2 - \lambda v\, h^3 - 2\lambda v\, h\phi^-\phi^+ - \lambda v\, h\phi_0^2 \\
& - \frac{\lambda}{4}h^4 - \frac{\lambda}{4}\phi_0^4 - \lambda(\phi^-\phi^+)^2 - \frac{\lambda}{2}h^2\phi_0^2 - \lambda\, h^2\phi^-\phi^+ - \lambda\,\phi_0^2\phi^-\phi^+
\end{aligned}$$

Among the terms above, the quadratic terms that determine the gauge fixing term can be identified as follows.

$$\begin{aligned}
& \left|\partial_\mu\phi^+ - \frac{i}{2}\left((2s_w c_w A'_\mu + (c_w^2 - s_w^2)Z'_\mu)\phi^+ + gW^+_\mu(v+h+i\phi^0)\right)\right|^2 \\
=& (\partial_\mu\phi^+)(\partial^\mu\phi^-) + \frac{i}{2}(2s_w c_w A'_\mu + (c_w^2 - s_w^2)Z'_\mu)(\phi^-\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^-) \\
& + \frac{i}{2}gv(W^-_\mu\partial_\mu\phi^+ - W^+_\mu\partial_\mu\phi^-) + \frac{i}{2}g(h+\phi^0)(W^-_\mu\partial_\mu\phi^+ - W^+_\mu\partial_\mu\phi^-) \\
& + \frac{1}{4}\left|2s_w c_w A'_\mu\phi^+ + (c_w^2 - s_w^2)Z'_\mu\phi^+ + g(v+h+i\phi^0)W^+_\mu\right|^2
\end{aligned} \tag{77}$$

First, to remove $\frac{igv}{2}(W^-_\mu\partial_\mu\phi^+ - W^+_\mu\partial_\mu\phi^-)$, we use the following gauge fixing term.

$$\begin{aligned}
\mathcal{L}_{gf} &= -\frac{1}{\xi}\left(\partial^\mu W^+_\mu - i\xi\frac{gv}{2}\phi^+\right)\left(\partial^\mu W^-_\mu + i\xi\frac{gv}{2}\phi^-\right) \\
&= -\frac{1}{\xi}|\partial^\mu W^+_\mu|^2 + \frac{igv}{2}(\phi^+\partial^\mu W^-_\mu - \phi^-\partial^\mu W^+_\mu) - \xi\left(\frac{gv}{2}\right)^2\phi^+\phi^-
\end{aligned} \tag{78}$$

Now, within the Feynman-'t Hooft gauge $\xi = 1$, we find ϕ^+, a Goldstone boson, acquires the same mass $\frac{gv}{2}$ as the W boson. Similarly, another quadratic term $\frac{\sqrt{g^2+g'^2}v}{2}(\partial_\mu\phi_0)Z_\mu$ is removed through an additional gauge fixing term $-\frac{1}{2\xi}(\partial_\mu Z_\mu - \frac{\sqrt{g^2+g'^2}v}{2}\xi\phi_0)^2$. Again, since $\xi = 1$, the mass of ϕ_0 is the same as that of the Z boson.

Example: $\phi^+(p_1)\phi^-(p_2) \to \phi^+(p_3)\phi^-(p_4)$

To see the power of the Goldstone equivalence theorem, let's find the $\phi^+(p_1)\phi^-(p_2) \to \phi^+(p_3)\phi^-(p_4)$ amplitude. The Feynman rules in general R_ξ gauge can be found from the Appendix B of Cheng & Li's textbook [2].

First, the s-channel amplitude mediated by A, Z bosons is as follows.[11]

$$\begin{aligned} i\mathcal{M} &= \left[(-ie)^2 \frac{-ig_{\mu\nu}}{s} + \left(-ig\frac{1-2\sin^2\theta_w}{2\cos\theta_w}\right)^2 \frac{-ig_{\mu\nu}}{s-m_Z^2}\right](p_2-p_1)_\mu(p_4-p_3)_\nu \\ &= i\left[\frac{e^2}{s} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{g^2}{s-m_Z^2}\right](p_1-p_2)\cdot(p_3-p_4) \end{aligned} \tag{79}$$

$$\begin{aligned} (p_1-p_2)\cdot(p_3-p_4) &= (p_1\cdot p_3 + p_2\cdot p_4) - (p_1\cdot p_4 + p_2\cdot p_3) \\ &= (2m_{\phi^\pm}^2 - t) - (2m_{\phi^\pm}^2 - u) = 4m_{\phi^\pm}^2 - s - 2t \end{aligned} \tag{80}$$

$$i\mathcal{M} \to -ig^2\left[\frac{\sin^2\theta_w}{s} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{1}{s-m_Z^2}\right](s+2t-4m_{\phi^\pm}^2) \tag{81}$$

The t-channel amplitude mediated by A, Z bosons can be obtained by exchanging p_2 and $-p_3$ from the above s-channel amplitude.

$$\begin{aligned} i\mathcal{M} &= \left[(-ie)^2 \frac{-ig_{\mu\nu}}{t} + \left(-ig\frac{1-2\sin^2\theta_w}{2\cos\theta_w}\right)^2 \frac{-ig_{\mu\nu}}{t-m_Z^2}\right](-p_3-p_1)_\mu(p_4+p_2)_\nu \\ &= -ig^2\left[\frac{\sin^2\theta_w}{t} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{1}{t-m_Z^2}\right](p_1+p_3)\cdot(p_2+p_4) \\ &= -ig^2\left[\frac{\sin^2\theta_w}{t} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{1}{t-m_Z^2}\right](2s+t-4m_{\phi^\pm}^2) \end{aligned} \tag{82}$$

The 4-point amplitude is simply as follows.

$$i\mathcal{M} = -i\frac{2m_h^2}{v^2} \tag{83}$$

Finally, the amplitude due to Higgs-exchange is as follows.

$$i\mathcal{M} = \left(-i\frac{m_h^2}{v}\right)^2\left(\frac{i}{s-m_h^2} + \frac{i}{t-m_h^2}\right) = -i\frac{m_h^4}{v^2}\left(\frac{1}{s-m_h^2} + \frac{1}{t-m_h^2}\right) \tag{84}$$

These steps are much easier than the evaluation of $W_L W_L \to W_L W_L$, as there are no more external polarization factors and cumbersome Lorentz contractions. Using the Fermi constant $G_F = \frac{1}{4\sqrt{2}}\frac{g^2}{m_W^2} = \frac{1}{\sqrt{2}v^2}$, the total amplitude of the $\phi^+(p_1)\phi^-(p_2) \to \phi^+(p_3)\phi^-(p_4)$ process can be summarized as follows.

$$\begin{aligned} \frac{1}{\sqrt{2}G_F}\mathcal{M} = &-4m_W^2\left[\frac{\sin^2\theta_w}{s} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{1}{s-m_Z^2}\right](s+2t-4m_{\phi^\pm}^2) \\ &- 4m_W^2\left[\frac{\sin^2\theta_w}{t} + \left(\frac{2\cos^2\theta_w-1}{2\cos\theta_w}\right)^2 \frac{1}{t-m_Z^2}\right](2s+t-4m_{\phi^\pm}^2) \\ &- 2m_h^2 - 2m_h^4\left(\frac{1}{s-m_h^2} + \frac{1}{t-m_h^2}\right) \end{aligned} \tag{85}$$

[11] Be careful that the propagator is different from that in the Unitary gauge! In a general R_ξ gauge, the propagator of a gauge boson is $i\Pi^{\mu\nu} = \frac{i}{p^2-m_V^2}\left(-\eta^{\mu\nu} + \frac{p^\mu p^\nu}{p^2-\xi m_V^2}(1-\xi)\right)$, so you must consider this when using the Goldstone equivalence theorem!

As we calculated for the Gauge bosons earlier, let's consider the High energy limit of the above amplitude.

$$
\begin{aligned}
\frac{1}{\sqrt{2}G_F}\mathcal{M} = &-4m_W^2\left[\frac{\sin^2\theta_w}{s} + \left(\frac{2\cos^2\theta_w - 1}{2\cos\theta_w}\right)^2\frac{1}{s}\left(1+\frac{m_Z^2}{s}+\cdots\right)\right](s+2t-4m_W^2) \\
&- 4m_W^2\left[\frac{\sin^2\theta_w}{t} + \left(\frac{2\cos^2\theta_w - 1}{2\cos\theta_w}\right)^2\frac{1}{t}\left(1+\frac{m_Z^2}{t}+\cdots\right)\right](2s+t-4m_W^2) \\
&- 2m_h^2 - 2m_h^4\left(\frac{1}{s}+\frac{1}{t}+\cdots\right) \\
&\approx -4m_W^2\left[\sin^2\theta_w + \left(\cos^2\theta_w - 1 + \frac{1}{4}\sec^2\theta_w\right)\left(1+\frac{m_Z^2}{s}+\cdots\right)\right]\frac{s+2t}{s} \\
&- 4m_W^2\left[\sin^2\theta_w + \left(\cos^2\theta_w - 1 + \frac{1}{4}\sec^2\theta_w\right)\left(1+\frac{m_Z^2}{t}+\cdots\right)\right]\frac{2s+t}{t} - 2m_h^2 \\
&\approx -m_W^2\sec^2\theta_w\left(\frac{s+2t}{s}+\frac{2s+t}{t}\right) - 2m_h^2 \\
\mathcal{M} \approx &-\sqrt{2}G_F\left(2m_Z^2\left(1+\frac{t}{s}+\frac{s}{t}\right)+2m_h^2\right) \qquad (86)
\end{aligned}
$$

This is identical to Eq. (68), the high energy limit $W_LW_L \to W_LW_L$ amplitude that we already calculated! This is the Goldstone equivalence theorem. For a theory where a gauge boson acquires mass through SSB, the high energy limit of an amplitude involving Longitudinal polarization is identical to substituting the gauge boson with its corresponding goldstone scalar at $\xi = 1$ gauge. Of course, for this convenience, there is the downside that you have to use a gauge other than $\xi \to \infty$, and you must keep the scalars corresponding to the goldstones.

Informal Derivation

The formal derivation of the Goldstone Equivalence theorem requires path integral formalism. The main idea of the proof is as follows. Consider the partition function (path integral) for gauge theory with SSB. Then, the contribution to partition function due to high-energy part of longitudinally polarized vector in $\xi \to \infty$ is almost equal to that of goldstone boson in $\xi \to 1$. So high energy goldstone boson can replace high energy, longitudinally polarized vector.

Let us show informal derivation. Consider S-matrix element involving single external massive gauge boson V in the initial state.[12]

$$\langle f|S|i, W_L\rangle \qquad (87)$$

This amplitude may be expressed by $\langle f|S|i\rangle$, the same but without W_L.[13]

$$\langle f|S|i, W_L\rangle = \epsilon^L\langle f|S\cdot a_L^\dagger(p)|i\rangle \qquad (88)$$

Now we use LSZ reduction formula.

$$\epsilon^L\langle f|S\cdot a_L^\dagger(p)|i\rangle = \epsilon^L i\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot V_\mu(x)|i\rangle \qquad (89)$$

[12] The following proof also can be done with gauge boson in the final state.

[13] The reader must beware that S-matrix element, and so amplitude $\mathcal{M}$, does not require energy momentum conservation. Such amplitude $\langle f|S|i\rangle$, however, does not contribute to physical observable because delta function for energy momentum conservation is involved during the steps evaluating them.

In the ultra-relativistic limit, the longitudinal polarization vector is essentially the same as p_μ.

$$\epsilon_\mu^L = \frac{p}{m_A}\left(1,0,0,\frac{E}{p}\right) = \frac{p}{m_A}\left(1,0,0,1+\frac{1}{2}\frac{m_A^2}{p^2}+\cdots\right) \approx \frac{p^\mu}{m_A} + \mathcal{O}\left(\frac{m_A^2}{p^2}\right)(0,\hat{\vec{p}}) \tag{90}$$

So at the high energy limit, we may replace ϵ^L by $\frac{p^\mu}{m_A}$.

$$\epsilon^L i\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot V_\mu(x)|i\rangle \approx \frac{ip^\mu}{m_V}\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot V_\mu(x)|i\rangle \tag{91}$$

The ip^μ factor can be absorbed into S matrix part.

$$\begin{aligned}\frac{ip^\mu}{m_V}\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot V_\mu(x)|i\rangle &= -\frac{1}{m_V}\int d^4x(\partial^\mu e^{-ip_V x})(\Box + m_V^2)\langle f|S\cdot V_\mu(x)|i\rangle \\ &= \frac{1}{m_V}\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot \partial^\mu V_\mu(x)|i\rangle \end{aligned}\tag{92}$$

By the way, the R_ξ gauge forces $\partial_\mu V^\mu = \xi m_V \phi$, where ϕ is the Goldstone boson to be eaten. So we can replace $\partial_\mu V^\mu$ by $\xi m_V \phi$.

$$\frac{1}{m_V}\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot \partial^\mu V_\mu(x)|i\rangle = \xi\int d^4x e^{-ip_V x}(\Box + m_V^2)\langle f|S\cdot\phi|i\rangle = \xi\langle f|S|i,\phi\rangle \tag{93}$$

The last step is just due to LSZ formula.

The whole step can be summarized as follows.

$$\langle f|S|i,W_L\rangle \approx \xi\langle f|S|i,\phi\rangle \rightarrow \mathcal{M}((i,W_L)\rightarrow f) \approx \xi\ \mathcal{M}((i,\phi)\rightarrow f) \quad \text{(High Energy Limit)} \tag{94}$$

So, the amplitude involving a massive gauge boson is equal to the same amplitude substituted by the corresponding Goldstone boson times ξ. In the Feynman-'t Hooft gauge, $\xi = 1$ so they are the same.

A Super Basics of Quantum Field Theory

For phenomenologists, the main purpose of learning quantum field theory(QFT) is probably to calculate amplitudes and the values of observables derived from them. In this appendix, I would like to briefly mention what QFT is roughly about, and some basic calculation methods.

Thinking about the Schrödinger equation, you can see that a particle (and a system) is described through a function that depends on position $\vec{x}$ and time t: the field $\psi(\vec{x},t)$. QFT is also a theory describing the behavior of fields. And it interprets the behavior of the field as the behavior of a particle. That is, QFT is ultimately just a theory about how to describe the behavior of particles (with quantum mechanical effects added).

I believe you know only limited kind of the Schrödinger equations can be solved exactly. Similarly, the QFT that can be solved exactly for a given potential are very limited. Just as we used perturbation theory for such cases in QM, QFT also assumes that the potential term (which we will call the interaction term from now on) can be treated as a perturbation, based on the exactly solvable case: the Free theory. Therefore, we first need to learn about the free theory in QFT.

The free theory describing a real number-valued field ϕ with mass m and spin (s) 0 is as follows.

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi = -\frac{1}{2}\phi(\partial_t^2 - \nabla^2 + m^2)\phi \tag{95}$$

The factor $1/2$ in front was determined so that the Hamiltonian value matches the energy (density) after Legendre transforming the Lagrangian (density) into the Hamiltonian (density).

A complex scalar field can be said to consist of two real scalar fields with the same mass: $\mathrm{Re}\ \phi$ and $\mathrm{Im}\ \phi$. $\phi = \frac{1}{\sqrt{2}}\left(\mathrm{Re}\ \phi + i\mathrm{Im}\ \phi\right)$ Its free theory is as follows.

$$\mathcal{L} = -\phi^*(\Box + m^2)\phi \tag{96}$$

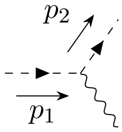
A vector-valued field V^μ has $s = 1$. The free theory describing a vector field with mass m is as follows.

$$\mathcal{L} = -\frac{1}{4}V_{\mu\nu}V^{\mu\nu} + \frac{1}{2}m^2 V_\mu V^\mu \equiv -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)(\partial^\mu V^\nu - \partial^\nu V^\mu) + \frac{1}{2}m^2 V_\mu V^\mu \tag{97}$$

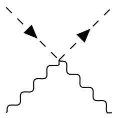
(As mentioned earlier) the interaction terms to be treated perturbatively consist of the product of three or more fields and a small coefficient. For example, in a theory called Scalar QED[14], the interaction terms are as follows.

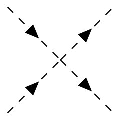
$$\mathcal{L}_{\text{int}} = ieA^\mu\left[\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right] + e^2\eta_{\mu\nu}A^\mu A^\nu\phi^*\phi - \frac{\lambda}{4}(\phi^*\phi)^2 \tag{98}$$

Each interaction term is interpreted as follows.

$$ieA^\mu\left[\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*\right] \rightarrow$$


$$= i \times ie \times (\pm ip_{1\mu} - \pm ip_{2\mu}) \tag{99}$$

$$e^2\eta_{\mu\nu}A^\mu A^\nu\phi^*\phi \rightarrow$$

$$= i \times e^2\eta_{\mu\nu} \tag{100}$$

$$-\frac{\lambda}{4}(\phi^*\phi)^2 \rightarrow$$

$$= i \times 4 \times \frac{\lambda}{4} \tag{101}$$

(The sign of $i \times ie \times (\pm ip_{1\mu} - \pm ip_{2\mu})$ is determined by whether it is ingoing or outgoing as shown at the very top.)

Now, for a given scattering process, the Matrix element can be calculated as follows.

1. First, draw points corresponding to the incoming particles and outgoing particles.
2. Each interaction term corresponds to a vertex where specific lines meet. Draw lines from the incoming particles and outgoing particles so that each line meets only at a vertex allowed by the interaction term.
3. The amplitude $i\mathcal{M}$ is the product of values corresponding to each vertex.
4. Lines that are not connected to any incoming or outgoing particle are propagators. The factors corresponding to these must also be multiplied in step 3 above.

The propagator of a scalar particle is $\frac{i}{p^2 - m^2}$. Here, p is the momentum that the scalar particle must have when considering momentum conservation. For the propagator of a vector particle, please refer to the main text.

B $2 \rightarrow n$

Most literature dealing with the unitarity of scattering amplitudes only covers $2 \rightarrow 2$ processes. For those of you who are not satisfied with this, here I will briefly talk about the unitarity of $2 \rightarrow n$ scattering amplitudes. When finding the Unitarity bound in the main text, by dealing with ultra-high-energy scattering,

[14] This theory (model) consists of a complex scalar field ϕ and a massless vector field A^μ. For each model, the particle contents and the form of their interactions are different, and we find and use the model suitable for a given physical situation.

we effectively treated all particles involved in the scattering as massless species. Similarly, here we will assume from the beginning that all external particles involved in the scattering are massless.

If we transform Eq. (8) into $2 \to n$, and assume a massless COM frame, $\sqrt{s} = p/2$, so we can obtain the following equation.

$$\mathrm{Im}\ \mathcal{M}(i \to i) = s\sigma_{\mathrm{tot}} = 16\pi \sum_{j=0}^{\infty}(2j+1)\mathrm{Im}\ a_j \geq s\sigma_{2\to n} \tag{102}$$

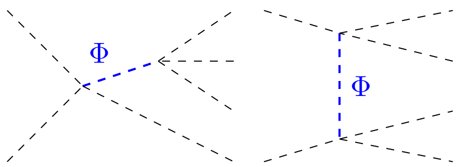
$\sigma_{2\to n}$ is as follows.

$$\sigma_{2\to n} = \frac{1}{8s}\int d\Pi\ (2\pi)^4\delta^4(\Sigma p)\mathcal{M}(2\to n) \tag{103}$$

$\mathcal{M}(2 \to n)$ is the matrix element associated with the $2 \to n$ process, which generally depends on the momenta of the outgoing particles.

At this stage, I would like to make an important simplification. Just as we only dealt with the scattering angle independent s-wave part (a_0) in $2 \to 2$ earlier, here we will also use the angle-independent part of the amplitude. This means we will insert an appropriate constant in place of $\mathcal{M}(2 \to n)$.

This approach is particularly appropriate for EFTs with an appropriate cutoff. For example, consider a $2 \to 4$ scattering in a theory where there is a heavy particle Φ with mass Λ and an interaction term $\phi^3\Phi$. The diagrams associated with this amplitude can be broadly divided into two types.



The amplitude at an energy scale lower than Λ turns out to be almost constant roughly as follows.

$$\mathcal{M} \sim \mathcal{O}(10)\frac{\lambda^2}{\Lambda^2} \tag{104}$$

($\mathcal{O}(10)$ is the number of independent diagrams.)

Of course, these amplitude is only justified when there is a particle heavier than working energy scale. Such scenario is well realized when dealing effective field theories that UV completion is unclear.

The above simplification also allows us to ignore the angle dependence of the scattering amplitude, resulting in $\mathcal{M} = 16\pi a_0$. Therefore, the inequality used previously: $|a_0| \leq 1$ still holds.

By applying the above simplification, the unitarity problem ultimately comes down to the problem of calculating the phase space integral V_n.

$$\sigma_{2\to n} = \frac{1}{8s}\int d\Pi\ (2\pi)^4\delta^4(\Sigma p)\mathcal{M}(2\to n) \to \frac{g}{8s}\int d\Pi\ (2\pi)^4\delta^4(\Sigma p) = \frac{g}{8s}\frac{V_n}{(2\pi)^{3n-4}} \tag{105}$$

The calculation process for V_n is as follows. [4]

$$V_n = \int \delta^4(\Sigma p)\prod_{i=1}^{n}\frac{d^3p_i}{2E_i} \tag{106}$$

First, the E_i in the denominator can be removed through the following process. (This equation holds even in the massive case.)

$$\int d^4p_i\delta(p_i^2 - m^2)\Theta((p^0)_i) = \int d^4p_i\delta((p^0)_i^2 - m^2 - \vec{p}_i^2)\Theta((p^0)_i)$$

$$= \int d^4p_i\left(\frac{\delta((p^0)_i - \sqrt{m^2+\vec{p}_i^2})}{2\sqrt{m^2+\vec{p}_i^2}} - \frac{\delta((p^0)_i + \sqrt{m^2+\vec{p}_i^2})}{2\sqrt{m^2+\vec{p}_i^2}}\right)\Theta((p^0)_i)$$

Here, the part where p^0 is negative is excluded according to the property of the Heaviside theta function. Introducing $E_i \equiv \sqrt{m^2 + \vec{p}_i^2}$,

$$\int d^4p_i\delta(p_i^2 - m^2)\Theta((p^0)_i) = \int d^3p_i dp_i^0 \frac{\delta((p^0)_i - E_i)}{2E_i}\Theta((p^0)_i) = \int \frac{d^3p_i}{2E_i}\Theta((p^0)_i) = \int \frac{d^3p_i}{2E_i}$$

Therefore, V_n can be described as follows.

$$\begin{aligned} V_n &= \int \delta^4(\Sigma p)\prod_{i=1}^{n} d^4p_i\delta(p_i^2 - m^2)\Theta((p^0)_i) = \int \delta^4(\Sigma p)\prod_{i=1}^{n} d^4p_i\ \delta(p_i^2 - m^2)\Theta((p^0)_i) \\ &= \int \delta^4(P - \Sigma_{i=1}^n p_i)\prod_{i=1}^{n} d^4p_i\ \delta(p_i^2)\Theta((p^0)_i) \quad (\text{massless}) \end{aligned} \tag{107}$$

This V_n can be interpreted as a value proportional to the number of ways n on-shell massless species can share the total 4-momentum P. V_n has a mass dimension of $2n - 4$. (There are n d^4k terms, so $4n$; $\delta((p^0)^2) = \frac{\delta(p^0)}{2p_0}$ gives -2 per term, and there are n of them, so $-2n$; momentum conservation $\delta^4(\Sigma p)$ gives -4.)

(Since we are dealing with the massless case) the only energy scale V_n has is the CoM energy $\sqrt{s}$. Thus, the form of V_n can be described as follows.

$$V_n = a_n s^{n-2} \tag{108}$$

Now, after finding a_2, we can find V_n by finding the relationship between a_n and a_{n+1}. V_2 is as follows.

$$\begin{aligned} V_2 &= \int d^4k_1 d^4k_2 \delta(k_1^2)\delta(k_2^2)\Theta(k_1^0)\Theta(k_2^0)\delta^4(P - k_1 - k_2) \\ &= \int_0^\infty dk_1^0 dk_2^0 \int_0^\infty dr_1 dr_2 \int_0^\pi d\theta_1 d\theta_2 \int_0^{2\pi} d\phi_1 d\phi_2 r_1^2 r_2^2 \sin\theta_1 \sin\theta_2 \delta(k_1^2)\delta(k_2^2)\delta^4(P - k_1 - k_2) \\ &= \int_0^\infty dk_1^0 dk_2^0 \int_0^\infty dr_1 dr_2 \int_0^\pi d\theta_1 d\theta_2 \int_0^{2\pi} d\phi_1 d\phi_2 \\ &\quad r_1^2 r_2^2 \sin\theta_1 \sin\theta_2 \frac{\delta(k_1 - r_1)\delta(k_1 - r_2)}{4r_1 r_2} \frac{\delta(P_0 - k_1^0 - k_2^0)\delta(r_1 - r_2)\delta(\theta_1 + \theta_2 - \pi)\delta(\phi_1 - \phi_2)}{r_1^2 \sin(\pi - \theta_2)} \end{aligned}$$

(For $\frac{\delta(r_1 - r_2)\delta(\theta_1 + \theta_2 - \pi)\delta(\phi_1 - \phi_2)}{r_1^2 \sin(\pi - \theta_2)}$, please refer to the Dirac delta function in spherical coordinates.) Since $\sin\theta = \sin(\pi - \theta)$,

$$\begin{aligned} V_2 =& \frac{1}{4}\int_0^\infty dk_1^0 dk_2^0 \int_0^\infty dr_1 dr_2 \int_0^\pi d\theta_1 d\theta_2 \int_0^{2\pi} d\phi_1 d\phi_2 \\ & \frac{r_2}{r_1}\sin\theta_1 \delta(k_1 - r_1)\delta(k_2 - r_2)\delta(P_0 - k_1^0 - k_2^0)\delta(r_1 - r_2)\delta(\theta_1 + \theta_2 - \pi)\delta(\phi_1 - \phi_2) \\ =& \frac{1}{4}\int_0^\infty dk_1^0 \delta(P_0 - 2k_1^0)\int_0^\pi d\theta_1 \sin\theta_1 \int_0^{2\pi} d\phi_1 = \pi \int_0^\infty dk_1^0 \delta(P_0 - 2k_1^0) = \frac{\pi}{2} \\ V_2 =& a_2 s^0 = \frac{\pi}{2} \ \rightarrow \ a_2 = \frac{\pi}{2} \end{aligned}$$

Now it is time to find the relationship between V_n and V_{n+1}. The relationship between the two comes from the fact that V_{n+1} is, in a sense, the integral of V_n.

$$V_{n+1} = \int d^4p_{n+1}\delta^4(P - p_{n+1} - \Sigma_{i=1}^n p_i)\delta(p_{n+1}^2)\Theta(p_{n+1}^0)\prod_{i=1}^{n} d^4p_i\ \delta(p_i^2)\Theta((p^0)_i)$$

For $P - p_{n+1} = Q$, $\int \delta^4(Q - \Sigma_{i=1}^n p_i) \prod_{i=1}^n d^4 p_i\ \delta(p_i^2)\Theta((p^0)_i) = V_n = a_n (Q^2)^{n-2}$. (Here, the Mandelstam $s = Q^2$) Since the total momentum P is a fixed value, substituting $p_{n+1} = P - Q$ gives,

$$V_{n+1} = \int d^4Q \delta((P-Q)^2)\Theta((P-Q)^0)V_n(Q^2)$$

$$= \int_{Q^0_{\min}}^{Q^0_{\max}} dQ^0 \int_{r_{\min}}^{r_{\max}} dr \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\ r^2 \sin\theta\ \delta((P-Q)^2)\Theta(P^0 - Q^0)V_n(Q^2)$$

With $Q = (Q_0, \vec{Q})$ (where $|\vec{Q}| = r$) and the COM momentum $P = (P^0, \vec{0})$, we have $(P-Q)^2 = (P^0 - Q^0)^2 - r^2$. Therefore,

$$V_{n+1} = 4\pi \int_{Q^0_{\min}}^{Q^0_{\max}} dQ^0 \int_{r_{\min}}^{r_{\max}} dr\ r^2\ \delta((P^0 - Q^0)^2 - r^2)\Theta(P^0 - Q^0)V_n(Q^2)$$

Substitute $\delta((P^0 - Q^0)^2 - r^2)\Theta(P^0 - Q^0) \to \frac{1}{2r}\delta((P^0 - Q^0) - r)$.

$$V_{n+1} = 2\pi \int_{Q^0_{\min}}^{Q^0_{\max}} dQ^0 \int_{r_{\min}}^{r_{\max}} dr\ r\delta((P^0 - Q^0) - r)V_n(Q^2)$$

Since the variable of V_n is $Q^2 = (Q^0)^2 - r^2$,

$$V_{n+1} = 2\pi \int_{Q^0_{\min}}^{Q^0_{\max}} dQ^0 \int_{r_{\min}}^{r_{\max}} dr\ r\ \delta(P^0 - Q^0 - r)V_n((Q^0)^2 - r^2) = 2\pi \int_{r_{\min}}^{r_{\max}} dr\ r\ V_n((P^0)^2 - 2P^0 r)$$
$$= 2\pi a_n \int_{r_{\min}}^{r_{\max}} dr\ r((P^0)^2 - 2P^0 r)^{n-2}$$

Regarding the possible range of r: $r_{\min}$ and $r_{\max}$ correspond to the magnitude of the 3-momentum of the $n+1$-th particle. The minimum value is 0, and the maximum value is $P^0/2$ when all the other particles go in the opposite direction.

$$V_{n+1} = 2\pi a_n \int_0^{P^0/2} dr\ r((P^0)^2 - 2P^0 r)^{n-2} = 2\pi a_n \frac{(P^0)^{2n-2}}{4n(n-1)} = \frac{\pi}{2n(n-1)} a_n s^{n-1} \tag{109}$$
$$= a_n \frac{\pi}{2n(n-1)} s^{(n+1)-2} = a_{n+1} s^{(n+1)-2} \tag{110}$$

Therefore, since $a_{n+1} = a_n \frac{\pi}{2n(n-1)}$ and $a_2 = \frac{\pi}{2}$,

$$V_n = a_n s^{n-2} = \frac{(\pi/2)^{n-1}}{\Gamma(n)\Gamma(n-1)} s^{n-2}$$

Thus, the following unitarity relation holds.

$$\sigma_{2\to n} \approx \frac{g}{8(2\pi)^{3n-4}} \frac{(\pi/2)^{n-1}}{\Gamma(n)\Gamma(n-1)} s^{n-3} = g\frac{4\pi^3}{(4\pi)^{2n}} \frac{s^{n-3}}{\Gamma(n)\Gamma(n-1)} \tag{111}$$
$$g\frac{4\pi^3}{(4\pi)^{2n}} \frac{s^{n-2}}{\Gamma(n)\Gamma(n-1)} \leq 16\pi\ \mathrm{Im}\ a_0 \leq 16\pi \tag{112}$$

The last part is because $|\mathrm{Im}\ a_0| \leq 1$.

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