Introduction of Cosmology

Beom Su Kim

JBNU, Jeonju 54896, Republic of Korea
`rlaqjatn0419@naver.com`

2026. 04. 18

Abstract

In research on high-energy theoretical physics, cosmological knowledge is often required. In this note, I aim to introduce the basic cosmological concepts needed for smooth research progress.

Contents

1 The Cosmological Principle and the FLRW Metric

1.1 The Cosmological Principle and the Origin of the FLRW Metric

In modern cosmology, one does not begin by assuming an arbitrary spacetime geometry. Instead, one imposes a symmetry principle motivated by observations on sufficiently large scales. This principle is the *cosmological principle*, which states that the universe is spatially homogeneous and isotropic when averaged over large enough distances.

Homogeneity means that there is no preferred spatial position. In other words, the large-scale properties of the universe are the same at every point. Isotropy means that there is no preferred spatial direction. That is, at a given point, the universe looks the same in every direction. These are logically distinct conditions: homogeneity concerns translations in space, whereas isotropy concerns rotations about a point. When both are imposed, the spatial geometry is highly constrained.

It is important to emphasize that the cosmological principle is not meant to hold exactly on small scales. On galactic or sub-galactic scales, the universe is obviously inhomogeneous, containing stars, galaxies, clusters, and voids. The principle is instead a statement about the large-scale averaged universe. Within general relativity, this means that each constant-time spatial hypersurface must be a maximally symmetric three-dimensional space.

A maximally symmetric three-dimensional space has constant spatial curvature. There are only three possibilities:

$$k = +1, \qquad k = 0, \qquad k = -1,$$

corresponding respectively to positive, zero, and negative spatial curvature. Thus, the spatial metric must be that of a constant-curvature three-space.

The next question is how time dependence may enter. The cosmological principle does not require the universe to be static. It only requires that at each cosmic time, the spatial slices remain homogeneous and isotropic. Therefore, the only allowed time dependence is an overall change of scale of the spatial geometry. This is described by the *scale factor* $a(t)$.

If γ_{ij} denotes the metric of a maximally symmetric three-dimensional space with constant curvature k, then the most general spatial metric compatible with homogeneity and isotropy is

$$g_{ij}(t, \mathbf{x}) = a^2(t)\, \gamma_{ij}(\mathbf{x}).$$

The factor appears as $a^2(t)$, rather than $a(t)$, because the metric measures squared distances.

To construct the full spacetime metric, one chooses the time coordinate t to be the proper time measured by comoving observers, namely observers at rest with respect to the cosmic fluid. With this choice, the spacetime interval takes the form

$$ds^2 = -dt^2 + a^2(t)\, \gamma_{ij} dx^i dx^j.$$

This is the general Friedmann–Lemaître–Robertson–Walker (FLRW) line element.

In spherical comoving coordinates, the metric of the maximally symmetric three-space can be written as

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

Hence the FLRW metric becomes

$$\boxed{ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right].}$$

The meaning of each term is clear. The coordinate t is the cosmic time, equal to the proper time of comoving observers. The function $a(t)$ describes the overall expansion or contraction of the universe. The parameter k specifies the sign of the spatial curvature. The coordinates (r, θ, ϕ) are comoving coordinates, meaning that objects moving only with the Hubble flow keep fixed spatial coordinates while their physical separations scale with $a(t)$.

Thus, the FLRW metric is not an arbitrary ansatz. It is the unique spacetime form compatible with the cosmological principle, once one chooses cosmic time as the proper time of comoving observers. In this sense, the logical chain is

$$\text{cosmological principle} \implies \text{maximally symmetric spatial slices} \implies \text{FLRW metric}.$$

1.2 Why the Scale Factor Appears Only in the Spatial Part

A common question is why the scale factor appears only in the spatial part of the FLRW metric,

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2,$$

rather than multiplying both time and space in some reciprocal form. The answer is that this is a consequence of the time coordinate choice.

In FLRW cosmology, one chooses t to be the proper time of comoving observers. For such observers, the spatial coordinates are fixed, so $dx^i = 0$, and the line element reduces to

$$ds^2 = -dt^2.$$

Hence

$$d\tau = dt.$$

This means that in these coordinates,

$$g_{tt} = -1.$$

Therefore, all information about cosmic expansion is placed into the spatial part of the metric, through the scale factor $a(t)$.

This does *not* mean that time is physically unaffected by the expansion while space is affected. Rather, it means that the time coordinate has been chosen to coincide with a physically meaningful proper time, namely the time measured by observers comoving with the cosmic fluid. With a different time coordinate, the metric can be written differently.

For example, if one defines the conformal time η by

$$d\eta = \frac{dt}{a(t)},$$

then the FLRW metric becomes

$$ds^2 = a^2(\eta)\left[-d\eta^2 + d\Sigma^2\right].$$

In this form, the same overall factor multiplies both the temporal and spatial parts. Thus, whether the scale factor appears only in the spatial sector or in both sectors is a matter of coordinate choice, not a difference in physics.

1.3 Why the Invariance of the Speed of Light is Not Violated

Another important question is whether the form

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2$$

conflicts with the invariance of the speed of light. The answer is no.

In general relativity, the statement that the speed of light is invariant does *not* mean that the coordinate speed of light must always be equal to c in every coordinate system. Rather, it means that in every local inertial frame, the speed of light measured by a nearby observer is always c.

To see this explicitly, consider the spatially flat case $k = 0$, for which

$$ds^2 = -dt^2 + a^2(t)\, dx^2$$

for radial motion in one spatial direction. A light ray follows a null curve, so $ds^2 = 0$. Therefore,

$$0 = -dt^2 + a^2(t)\, dx^2,$$

which implies

$$\frac{dx}{dt} = \pm \frac{1}{a(t)}.$$

Thus, the coordinate speed of light with respect to the comoving coordinate x is not constant; it depends on the scale factor.

However, x is not a physical distance coordinate. The physical spatial interval at fixed cosmic time is

$$dl = a(t)\, dx.$$

Hence, along a null trajectory,

$$\frac{dl}{dt} = a(t)\frac{dx}{dt} = a(t) \cdot \frac{1}{a(t)} = 1$$

in units where $c = 1$. Therefore, the locally measured physical speed of light remains constant.

This is the essential distinction:

$$\boxed{\text{the coordinate speed of light may vary, but the locally measured physical speed of light is always } c.}$$

The reason is that general relativity preserves the null structure of spacetime. Light always propagates along null curves satisfying

$$ds^2 = 0.$$

This condition determines the local light cone structure. In any sufficiently small neighborhood of spacetime, one can introduce local inertial coordinates in which the metric is approximately Minkowskian,

$$ds^2 \approx -dT^2 + dX^2 + dY^2 + dZ^2,$$

and in such coordinates light always travels with speed c.

1.4 Cosmological Redshift in an FLRW Universe

The expansion of the universe also explains the cosmological redshift. The key result is

$$\boxed{1 + z = \frac{a_0}{a_e},}$$

where a_e is the scale factor at emission, a_0 is the scale factor at observation, and z is the redshift.

The redshift is defined by

$$1 + z \equiv \frac{\lambda_0}{\lambda_e},$$

where λ_e is the emitted wavelength and λ_0 is the observed wavelength.

To derive this relation, consider a radial light ray in the FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\frac{dr^2}{1 - kr^2}.$$

For a null trajectory,

$$ds^2 = 0,$$

so

$$\frac{dt}{a(t)} = \pm \frac{dr}{\sqrt{1 - kr^2}}.$$

Now consider two successive wave crests emitted by the same comoving source. Let the first crest be emitted at t_e and observed at t_0, and let the second crest be emitted at $t_e + \delta t_e$ and observed at $t_0 + \delta t_0$. Since both crests travel along the same comoving spatial path, the integrals must satisfy

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_{t_e+\delta t_e}^{t_0+\delta t_0} \frac{dt}{a(t)}.$$

Subtracting the two expressions gives

$$\int_{t_e}^{t_e+\delta t_e} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{dt}{a(t)}.$$

If the intervals are sufficiently small, the scale factor may be treated as approximately constant over each interval, so one obtains

$$\frac{\delta t_e}{a_e} = \frac{\delta t_0}{a_0}.$$

Hence,

$$\frac{\delta t_0}{\delta t_e} = \frac{a_0}{a_e}.$$

Since the period of the wave is the time interval between successive crests, and since wavelength is proportional to period,

$$\frac{\lambda_0}{\lambda_e} = \frac{\delta t_0}{\delta t_e} = \frac{a_0}{a_e}.$$

Therefore,

$$\boxed{1 + z = \frac{a_0}{a_e}.}$$

Equivalently, since frequency is the inverse of the period,

$$\nu \propto \frac{1}{a}.$$

Thus, as the universe expands, the wavelength of light stretches while the frequency decreases.

This effect should not be interpreted merely as an ordinary special-relativistic Doppler shift. For nearby galaxies, the distinction is small, and one may use a Doppler-like interpretation approximately. But for cosmology in general, the redshift is more fundamentally understood as the stretching of wavelengths by the expansion of the FLRW spacetime itself.

It is customary to normalize the present scale factor to unity,

$$a_0 = 1.$$

Then the redshift relation becomes

$$a_e = \frac{1}{1+z}.$$

Thus, a larger redshift corresponds to a smaller scale factor at emission, meaning that highly redshifted light was emitted when the universe was younger and smaller.

1.4.1 Derivation of $E \propto a^{-1}$ from the Geodesic Equation

A photon moves along a null geodesic, so its four-momentum is

$$p^\mu \equiv \frac{dx^\mu}{d\lambda}, \tag{1}$$

where λ is an affine parameter, and the geodesic equation is

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} p^\alpha p^\beta = 0. \tag{2}$$

To determine how the photon energy evolves, we consider the $\mu = 0$ component:

$$\frac{dp^0}{d\lambda} + \Gamma^0_{\alpha\beta} p^\alpha p^\beta = 0. \tag{3}$$

For the FLRW metric with signature $(-+++)$,

$$g_{00} = -1, \qquad g_{ij} = a^2(t)\gamma_{ij}, \tag{4}$$

and the relevant Christoffel symbol is

$$\Gamma^0_{ij} = \frac{1}{2} g^{00} \left(-\partial_0 g_{ij}\right). \tag{5}$$

Since $g^{00} = -1$ and

$$\partial_0 g_{ij} = 2a\dot{a}\,\gamma_{ij}, \tag{6}$$

we obtain

$$\Gamma^0_{ij} = a\dot{a}\,\gamma_{ij}. \tag{7}$$

Also, $\Gamma^0_{00} = 0$ and $\Gamma^0_{0i} = 0$, so the time component of the geodesic equation reduces to

$$\frac{dp^0}{d\lambda} + a\dot{a}\,\gamma_{ij} p^i p^j = 0. \tag{8}$$

Now we use the null condition for the photon trajectory:

$$g_{\mu\nu} p^\mu p^\nu = 0. \tag{9}$$

Thus,

$$-(p^0)^2 + a^2 \gamma_{ij} p^i p^j = 0, \tag{10}$$

which implies

$$a^2 \gamma_{ij} p^i p^j = (p^0)^2. \tag{11}$$

Therefore,

$$\gamma_{ij} p^i p^j = \frac{(p^0)^2}{a^2}. \tag{12}$$

Substituting this into the geodesic equation gives

$$\frac{dp^0}{d\lambda} + a\dot{a} \cdot \frac{(p^0)^2}{a^2} = 0, \tag{13}$$

or

$$\frac{dp^0}{d\lambda} + \frac{\dot{a}}{a}(p^0)^2 = 0. \tag{14}$$

Using

$$p^0 = \frac{dt}{d\lambda}, \tag{15}$$

we can rewrite

$$\frac{dp^0}{d\lambda} = \frac{dp^0}{dt}\frac{dt}{d\lambda} = p^0 \frac{dp^0}{dt}. \tag{16}$$

Hence,

$$p^0 \frac{dp^0}{dt} + \frac{\dot{a}}{a}(p^0)^2 = 0. \tag{17}$$

Dividing by $p^0 \neq 0$, we obtain

$$\frac{dp^0}{dt} + \frac{\dot{a}}{a} p^0 = 0. \tag{18}$$

This may be written as

$$\frac{1}{p^0}\frac{dp^0}{dt} = -\frac{\dot{a}}{a}. \tag{19}$$

Integrating both sides yields

$$\ln p^0 = -\ln a + \text{constant}, \tag{20}$$

so that

$$p^0 \propto a^{-1}. \tag{21}$$

Finally, the photon energy measured by a comoving observer is

$$E = -p_\mu u^\mu, \tag{22}$$

where the four-velocity of the comoving observer is

$$u^\mu = (1, 0, 0, 0). \tag{23}$$

Since

$$p_0 = g_{00} p^0 = -p^0, \tag{24}$$

it follows that

$$E = -p_0 u^0 = p^0. \tag{25}$$

Therefore,

$$\boxed{E \propto a^{-1}.} \tag{26}$$

Thus, the energy of radiation decreases inversely with the scale factor as the universe expands. This is the geodesic origin of cosmological redshift.

1.5 Summary

The overall logical structure is as follows. The cosmological principle states that the universe is spatially homogeneous and isotropic on large scales. These symmetry requirements imply that each spatial slice of constant cosmic time must be a maximally symmetric three-dimensional space of constant curvature. The only allowed time dependence consistent with those symmetries is a universal scale factor $a(t)$, leading uniquely to the FLRW metric,

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j.$$

The fact that the scale factor appears only in the spatial part is a consequence of choosing cosmic time as the proper time of comoving observers. This does not violate the invariance of the speed of light, because in general relativity the physically meaningful statement is that the speed of light is always c in local inertial frames, not that its coordinate speed must be the same in every coordinate system.

Finally, the null propagation of light in an FLRW spacetime implies that successive wave crests are stretched in proportion to the expansion of the universe. This leads directly to the cosmological redshift relation

$$1 + z = \frac{a_0}{a_e}.$$

Thus, redshift is a direct geometrical consequence of propagation through an expanding FLRW universe.

2 Hilbert Action

To understand the significance of the Hilbert action, it is useful to distinguish clearly between the *historical development* of general relativity and the *modern logical formulation* of the theory.

2.1 Historical Development

At the end of the nineteenth century, gravity was successfully described by Newtonian theory. However, after the advent of special relativity, it became clear that Newtonian gravity could not be the final theory of gravitation. Newtonian gravity assumes absolute time, and its instantaneous action-at-a-distance structure is not naturally compatible with relativistic causality.

Einstein's path toward general relativity began with the equivalence principle. In its essential form, the equivalence principle states that, locally, the effects of a gravitational field are indistinguishable from those of acceleration. This idea suggested that gravity should not be understood merely as an ordinary force, but rather as something deeply connected with the structure of spacetime itself.

From about 1907 to 1915, Einstein worked to construct a relativistic theory of gravitation. Several major conceptual steps were required. First, he had to identify the basic gravitational variable. Eventually, this became the spacetime metric $g_{\mu\nu}$. Second, he had to find the appropriate mathematical language, which led to the adoption of Riemannian geometry, tensor calculus, and the curvature of spacetime. Third, and most importantly, he had to determine the field equations satisfied by the metric.

Thus, Einstein's central problem was the following: if gravity is encoded in the metric $g_{\mu\nu}$, what equation determines the metric itself?

The decisive developments occurred in 1915. Einstein presented a sequence of papers in November 1915 that led to the final form of the gravitational field equations. At nearly the same time, David Hilbert was also working on a fundamental formulation of gravitation. Hilbert approached the problem from a more mathematical point of view, asking whether the theory of gravitation could be formulated by means of a variational principle.

It is therefore historically inaccurate to say, in an oversimplified way, that Hilbert merely copied Einstein, or that Einstein merely adopted Hilbert's formalism after the fact. A more accurate statement is that Einstein's search for the gravitational field equations and Hilbert's formulation of the theory in terms of an action principle took place almost simultaneously and in mutual interaction.

One may summarize the historical distinction as follows:

$$\text{Einstein's main focus: gravitational field equations,}$$

$$\text{Hilbert's main focus: the action principle underlying those equations.}$$

2.2 The Modern Viewpoint

In modern theoretical physics, the action principle is usually regarded as more fundamental than the field equations themselves. One first specifies an action functional, and then obtains the equations of motion by requiring the action to be stationary under variations of the dynamical variables.

In classical mechanics, for a generalized coordinate $q(t)$, one writes

$$S[q] = \int L(q, \dot{q}, t)\, dt,$$

and the condition

$$\delta S = 0$$

leads to the Euler–Lagrange equation.

Field theory works in exactly the same way. For a field $\phi(x)$, one writes an action of the form

$$S[\phi] = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi),$$

and variation of the action gives the field equation.

General relativity follows the same principle, but the dynamical variable is no longer a particle trajectory or an ordinary field on spacetime. Instead, the fundamental variable is the spacetime metric itself:

$$g_{\mu\nu}(x).$$

Since the metric determines distances, times, causal structure, and curvature, it is the natural candidate for the fundamental gravitational field.

Thus, in general relativity, one seeks an action of the form

$$S[g_{\mu\nu}, \psi],$$

where $g_{\mu\nu}$ is the metric and ψ denotes the matter fields. The field equations for gravity are then obtained by varying the action with respect to $g_{\mu\nu}$.

2.2.1 Why the Gravitational Action Has the Form $\sqrt{-g}R$

To construct the gravitational action, one must satisfy several requirements.

First, the theory must be generally covariant. That is, the action must be invariant under coordinate transformations. Therefore, the action must be written as an integral of a scalar density over spacetime.

Second, on a curved spacetime, the natural invariant volume element is not simply d^4x, but rather

$$\sqrt{-g}\, d^4x,$$

where

$$g = \det(g_{\mu\nu}).$$

The factor $\sqrt{-g}$ is the curved-spacetime analogue of the Jacobian that appears when one changes coordinates in ordinary multivariable integration.

Third, the integrand must be constructed from the metric and its derivatives. Since gravity is identified with spacetime curvature, the action must involve a scalar built from curvature. The simplest such scalar is the Ricci scalar

$$R = g^{\mu\nu} R_{\mu\nu}.$$

Therefore, the simplest generally covariant gravitational action is

$$\boxed{S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x\, \sqrt{-g}\, R.}$$

This is the Einstein–Hilbert action.

If one includes the cosmological constant, the gravitational part becomes

$$\boxed{S_{\text{grav}} = \frac{1}{16\pi G} \int d^4x\, \sqrt{-g}\, (R - 2\Lambda).}$$

Including matter fields ψ, the total action is

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,(R-2\Lambda) + S_m[g,\psi].}$$

2.2.2 Meaning of the Statement: "The Hilbert Action Has the Gravitational Field Equation as Its Equation of Motion"

This statement is precise, but it must be interpreted correctly.

In mechanics, the action does not itself represent the equation of motion. Rather, the equation of motion is obtained from the condition that the action be stationary under variations of the dynamical variable. For example,

$$\delta S[q] = 0$$

gives the Euler–Lagrange equation for $q(t)$.

Exactly the same logic applies in general relativity. The dynamical variable is now the metric $g_{\mu\nu}$, so one imposes

$$\delta S[g_{\mu\nu},\psi] = 0$$

under variations of the metric. The result is the gravitational field equation.

Thus, when one says that the Hilbert action has the gravitational field equation as its equation of motion, one means:

$$\boxed{\text{The Einstein field equation is obtained by varying the Hilbert action with respect to the metric.}}$$

In modern notation, the variation of the total action leads to

$$\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu},}$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$$

is the Einstein tensor, and $T_{\mu\nu}$ is the energy–momentum tensor derived from the matter action.

The phrase "equation of motion" here should not be misunderstood as referring only to particle motion. In field theory, the term refers more generally to the dynamical equations satisfied by the fields. Thus, the Einstein field equation is the equation of motion of the metric field $g_{\mu\nu}$.

2.2.3 Why the Hilbert Action is Special

The Einstein–Hilbert action is distinguished by several important properties.

First, it is the simplest scalar action that can be constructed from the metric and curvature while maintaining general covariance.

Second, its variation with respect to the metric produces the Einstein field equation, which is the correct relativistic field equation for gravitation.

Third, in the weak-field and low-velocity limit, the resulting theory reproduces Newtonian gravity.

For these reasons, the Hilbert action is not merely a compact rewriting of the Einstein equation. Rather, it is the variational principle from which the Einstein field equation follows.

2.3 Derivation of the Einstein Field Equations from the Einstein–Hilbert Action

We now derive the gravitational field equations from the principle of stationary action. The starting point is the total action for gravity, matter, and radiation:

$$S[g,\psi] = S_{\mathrm{EH}}[g] + S_m[g,\psi], \tag{27}$$

where

$$S_{\mathrm{EH}}[g] = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,(R-2\Lambda), \tag{28}$$

and

$$S_m[g,\psi] = \int_{\mathcal{M}} d^4x\,\sqrt{-g}\,\mathcal{L}_m(g,\psi,\nabla\psi). \tag{29}$$

Here $g_{\mu\nu}$ is the spacetime metric, $g = \det(g_{\mu\nu})$, R is the Ricci scalar, Λ is the cosmological constant, and ψ collectively denotes the matter and radiation fields.

Our goal is to vary the total action with respect to the metric and impose the stationary action condition

$$\delta S = 0. \tag{30}$$

This yields the Euler–Lagrange equation for the metric field, namely the Einstein field equation.

2.3.1 Variation of the Einstein–Hilbert Action

We first vary the gravitational part:

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\delta\left[\sqrt{-g}(R-2\Lambda)\right]. \tag{31}$$

Expanding the variation gives

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\left[\delta(\sqrt{-g}R) - 2\Lambda\,\delta\sqrt{-g}\right]. \tag{32}$$

2.3.2 Variation of $\sqrt{-g}$

A standard result is

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}. \tag{33}$$

Therefore, the cosmological constant term varies as

$$-2\Lambda\,\delta\sqrt{-g} = \sqrt{-g}\,\Lambda g_{\mu\nu}\,\delta g^{\mu\nu}. \tag{34}$$

2.3.3 Variation of the Ricci scalar

Since

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{35}$$

its variation is

$$\delta R = R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}. \tag{36}$$

Hence,

$$\begin{aligned}
\delta(\sqrt{-g}R) &= R\,\delta\sqrt{-g} + \sqrt{-g}\,\delta R &(37)\\
&= R\left(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}\right) + \sqrt{-g}\,(R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}) &(38)\\
&= \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}. &(39)
\end{aligned}$$

Thus the variation of the Einstein–Hilbert action becomes

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \right]. \tag{40}$$

2.3.4 Boundary term and boundary condition

The remaining term $g^{\mu\nu}\delta R_{\mu\nu}$ can be rewritten using the Palatini identity:

$$\delta R_{\mu\nu} = \nabla_\rho (\delta \Gamma^\rho_{\mu\nu}) - \nabla_\nu (\delta \Gamma^\rho_{\mu\rho}). \tag{41}$$

Therefore,

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\rho V^\rho \tag{42}$$

for some vector V^ρ. Hence,

$$\int_{\mathcal{M}} d^4x \sqrt{-g}\, g^{\mu\nu} \delta R_{\mu\nu} = \int_{\partial\mathcal{M}} d^3y \sqrt{|h|}\, n_\rho V^\rho, \tag{43}$$

which is a pure boundary term.

This boundary contribution does not vanish automatically. To obtain a well-defined variational principle with fixed induced metric on the boundary,

$$\delta h_{ab}\big|_{\partial\mathcal{M}} = 0, \tag{44}$$

one supplements the Einstein–Hilbert action with the Gibbons–Hawking–York boundary term. Under this Dirichlet boundary condition, the total boundary variation vanishes, and only the bulk term remains.

Accordingly, the bulk variation of the gravitational action is

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \right) \delta g^{\mu\nu}. \tag{45}$$

Introducing the Einstein tensor,

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{46}$$

this becomes

$$\delta S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g}\, (G_{\mu\nu} + \Lambda g_{\mu\nu})\, \delta g^{\mu\nu}. \tag{47}$$

2.3.5 Variation of the Matter Action

The matter and radiation action is

$$S_m[g, \psi] = \int_{\mathcal{M}} d^4x \sqrt{-g}\, \mathcal{L}_m. \tag{48}$$

Its variation with respect to the metric defines the stress–energy tensor:

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}}. \tag{49}$$

Equivalently,

$$\delta S_m = -\frac{1}{2} \int_{\mathcal{M}} d^4x \sqrt{-g}\, T_{\mu\nu}\, \delta g^{\mu\nu}. \tag{50}$$

2.3.6 Field Equation from the Stationary Action Principle

The variation of the total action is therefore

$$\delta S = \delta S_{\text{EH}} + \delta S_m. \tag{51}$$

Substituting the two results gives

$$\delta S = \frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,(G_{\mu\nu}+\Lambda g_{\mu\nu})\,\delta g^{\mu\nu} - \frac{1}{2}\int_{\mathcal{M}} d^4x\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu} \tag{52}$$

$$= \int_{\mathcal{M}} d^4x\,\sqrt{-g}\left[\frac{1}{16\pi G}(G_{\mu\nu}+\Lambda g_{\mu\nu}) - \frac{1}{2}T_{\mu\nu}\right]\delta g^{\mu\nu}. \tag{53}$$

Since the metric variation $\delta g^{\mu\nu}$ is arbitrary in the bulk, the stationary action condition

$$\delta S = 0 \tag{54}$$

implies

$$\frac{1}{16\pi G}(G_{\mu\nu}+\Lambda g_{\mu\nu}) - \frac{1}{2}T_{\mu\nu} = 0. \tag{55}$$

Multiplying by $16\pi G$, we obtain the Einstein field equation:

$$\boxed{G_{\mu\nu}+\Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu}.} \tag{56}$$

Using the definition of the Einstein tensor, this may also be written as

$$\boxed{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\,T_{\mu\nu}.} \tag{57}$$

Thus, the Einstein field equation is the Euler–Lagrange equation obtained by varying the total action with respect to the metric tensor.

2.4 Summary

Historically, Einstein was primarily searching for the correct gravitational field equation, while Hilbert formulated the theory in a variational framework. These developments occurred almost simultaneously and in close interaction.

From the modern point of view, however, the action principle is more fundamental. The Einstein–Hilbert action,

$$\boxed{S_{\text{EH}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,R,}$$

or, more generally,

$$\boxed{S = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}(R-2\Lambda) + S_m[g,\psi],}$$

is the action whose variation with respect to the metric yields the Einstein field equation. In this sense, the Hilbert action is the action whose Euler–Lagrange equation is precisely the gravitational field equation of general relativity.

3 Derivation of the Friedmann Equations from the Einstein Field Equations

3.1 Structure of the Field Equations

The Einstein field equations, in natural units, may be written as

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi G T_{\mu\nu}. \tag{58}$$

After Einstein realized that this equation, when applied to a homogeneous and isotropic universe, allows for an expanding or contracting cosmos, he introduced the cosmological constant term. The modified equation becomes

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{59}$$

Here, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $T_{\mu\nu}$ is the stress–energy tensor, and Λ is the cosmological constant. The physical meaning of the field equation is that the distribution of matter and energy determines the curvature of spacetime.

3.2 Left-Hand Side of the Field Equation

3.2.1 Metric Tensor

To solve the field equations, one must begin with the metric tensor, which encodes the distances and angles of spacetime. Assuming a homogeneous and isotropic universe, the line element is

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2) \right]. \tag{60}$$

This is the FLRW metric, named after Friedmann, Lemaître, Robertson, and Walker.

The parameter k represents the spatial curvature of the universe. Although it may in principle take various values, the physically meaningful cases are

$$k = -1, \qquad k = 0, \qquad k = 1,$$

corresponding to open, flat, and closed spatial geometries, respectively.

In matrix form, the metric is

$$g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & \dfrac{a^2(t)}{1 - kr^2} & 0 & 0 \\ 0 & 0 & a^2(t) r^2 & 0 \\ 0 & 0 & 0 & a^2(t) r^2 \sin^2\theta \end{bmatrix}. \tag{61}$$

The nonvanishing covariant components are therefore

$$g_{tt} = -1, \qquad g_{rr} = \frac{a^2(t)}{1 - kr^2}, \qquad g_{\theta\theta} = a^2(t) r^2, \qquad g_{\phi\phi} = a^2(t) r^2 \sin^2\theta. \tag{62}$$

The corresponding contravariant components are

$$g^{tt} = -1, \qquad g^{rr} = \frac{1 - kr^2}{a^2(t)}, \qquad g^{\theta\theta} = \frac{1}{a^2(t) r^2}, \qquad g^{\phi\phi} = \frac{1}{a^2(t) r^2 \sin^2\theta}. \tag{63}$$

3.2.2 Christoffel Symbols

The Christoffel symbols describe how the metric changes across spacetime and play the role of the spacetime connection. They are defined by

$$\Gamma^{\ell}{}_{ji} = \frac{1}{2} g^{\ell m} \left(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij} \right). \tag{64}$$

Using the FLRW metric, the nonvanishing Christoffel symbols are

$$\Gamma^{t}{}_{rr} = \frac{a\dot{a}}{1 - kr^2}, \tag{65}$$

$$\Gamma^{t}{}_{\theta\theta} = a\dot{a} \, r^2, \tag{66}$$

$$\Gamma^{t}{}_{\phi\phi} = a\dot{a} \, r^2 \sin^2\theta, \tag{67}$$

$$\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \frac{\dot{a}}{a}, \tag{68}$$

$$\Gamma^r{}_{rr} = \frac{kr}{1-kr^2}, \tag{69}$$

$$\Gamma^r{}_{\theta\theta} = -r(1-kr^2), \tag{70}$$

$$\Gamma^r{}_{\phi\phi} = -r(1-kr^2)\sin^2\theta, \tag{71}$$

$$\Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = \frac{1}{r}, \tag{72}$$

$$\Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta, \tag{73}$$

$$\Gamma^\phi{}_{\phi\theta} = \Gamma^\phi{}_{\theta\phi} = \cot\theta. \tag{74}$$

3.2.3 Riemann Curvature Tensor

Although the Christoffel symbols contain information about the curvature of spacetime, they are not tensors. From them, one constructs the Riemann curvature tensor,

$$R^\ell{}_{kji} = \partial_i\Gamma^\ell{}_{kj} - \partial_j\Gamma^\ell{}_{ki} + \Gamma^m{}_{kj}\Gamma^\ell{}_{mi} - \Gamma^m{}_{ki}\Gamma^\ell{}_{mj}. \tag{75}$$

The Riemann tensor measures how curved spacetime is at each point.

3.2.4 Ricci Tensor and Ricci Scalar

The quantities that enter the Einstein field equations are not the full Riemann tensor, but its contractions: the Ricci tensor and the Ricci scalar. The Ricci tensor is defined by

$$R_{\alpha\beta} = R^\lambda{}_{\alpha\lambda\beta}, \tag{76}$$

and the Ricci scalar by

$$R = g^{\alpha\beta}R_{\alpha\beta}. \tag{77}$$

For the FLRW metric, the nonvanishing Ricci tensor components are

$$R_{tt} = -3\frac{\ddot{a}}{a}, \tag{78}$$

$$R_{rr} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}, \tag{79}$$

$$R_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k), \tag{80}$$

$$R_{\phi\phi} = r^2\sin^2\theta\,(a\ddot{a} + 2\dot{a}^2 + 2k). \tag{81}$$

The Ricci scalar is then obtained by contraction:

$$R = g^{tt}R_{tt} + g^{rr}R_{rr} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi}. \tag{82}$$

3.2.5 Einstein Tensor

Finally, one constructs the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{83}$$

This tensor represents the geometric side of the field equations.

3.3 Right-Hand Side of the Field Equation

3.3.1 Stress–Energy Tensor of a Perfect Fluid

Once the geometric side has been determined, the next step is to specify the distribution of matter and energy. On cosmological scales, the dominant sources are well approximated by an idealized perfect fluid. The stress–energy tensor of a perfect fluid is

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\, g_{\mu\nu}, \tag{84}$$

where ρ is the energy density, p is the pressure, and u^μ is the four-velocity of the fluid.

3.4 FLRW Solution: The Friedmann Equations

Now that both sides of the Einstein field equation have been specified, the Friedmann equations can be derived.

3.4.1 First Friedmann Equation

For the (t,t) component, the field equation is

$$R_{tt} - \frac{1}{2}R g_{tt} = 8\pi G T_{tt} - \Lambda g_{tt}. \tag{85}$$

Substituting the FLRW expressions gives

$$-3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + 3\frac{k}{a^2} = 8\pi G\,\rho + \Lambda. \tag{86}$$

Rearranging, one obtains

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}} \tag{87}$$

which is the first Friedmann equation.

The quantity

$$H \equiv \frac{\dot{a}}{a} \tag{88}$$

is called the Hubble parameter. Its present-day value is the Hubble constant.

3.4.2 Second Friedmann Equation

For the spatial components, one finds

$$-\frac{g_{ii}}{a^2}(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{1}{2}R g_{ii} = 8\pi G(-p)g_{ii} - \Lambda g_{ii}. \tag{89}$$

Substituting the curvature terms and simplifying yields

$$-\frac{\ddot{a}}{a} - 2\left(\frac{\dot{a}}{a}\right)^2 - \frac{2k}{a^2} + 3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} = -8\pi G p + \Lambda. \tag{90}$$

This reduces to

$$\frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = -4\pi G p + \frac{\Lambda}{2} - \frac{1}{2}\frac{k}{a^2}. \tag{91}$$

Using the first Friedmann equation to eliminate H^2, one obtains the second Friedmann equation:

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}} \tag{92}$$

This equation describes the acceleration of the scale factor.

3.5 Interpretation

If the cosmological constant is absent, then the second Friedmann equation implies that a static universe is not generically possible: the universe must either expand or contract. This was one of the reasons Einstein introduced the cosmological constant into the field equations.
Historically, Einstein later regarded the cosmological constant as a mistake after Hubble's discovery of cosmic expansion. Ironically, in modern cosmology the cosmological constant has returned as one of the leading candidates for dark energy, which drives the present accelerated expansion of the universe.

4 Energy components of our universe-Radiation, Matter, and the Cosmological Constant

In cosmology, the dominant energy components of the universe are usually classified into three idealized sectors: radiation, nonrelativistic matter, and the cosmological constant. Their equation-of-state parameters are defined through the relation

$$p = w\rho, \tag{93}$$

where ρ is the energy density, p is the pressure, and w is the equation-of-state parameter.

The starting point in general relativity is the definition of the stress–energy tensor from the matter action:

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}} \tag{94}$$

or, equivalently,

$$\boxed{\delta S_m = -\frac{1}{2}\int d^4x\,\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu}.} \tag{95}$$

On cosmological scales, each component is modeled as a homogeneous and isotropic perfect fluid. Therefore its stress–energy tensor must take the form

$$\boxed{T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p\,g_{\mu\nu},} \tag{96}$$

where u^μ is the four-velocity of the fluid. In the fluid rest frame, this becomes

$$T^\mu{}_\nu = \mathrm{diag}(-\rho,\, p,\, p,\, p) \tag{97}$$

for the metric signature $(-,+,+,+)$. Hence, once the form of $T_{\mu\nu}$ is fixed by the physical definition of each component, the corresponding pressure and equation-of-state parameter can be read off immediately.

4.1 Radiation: $w = \frac{1}{3}$

Radiation is defined as a relativistic and isotropic distribution of particles. For such a distribution, the particle energy satisfies

$$E = |\mathbf{p}|, \tag{98}$$

where $\mathbf{p}$ is the spatial momentum. In kinetic theory, the stress–energy tensor of an isotropic particle distribution may be written as

$$T^{\mu\nu} = \int \frac{d^3p}{E}\,p^\mu p^\nu f(\mathbf{p}), \tag{99}$$

where $f(\mathbf{p})$ is the isotropic distribution function.

The energy density is

$$\rho = T^{00} = \int d^3p\,E\,f(\mathbf{p}), \tag{100}$$

whereas the pressure is given by one spatial diagonal component,

$$p = T^{11} = \int \frac{d^3p}{E}\, p_1^2 f(\mathbf{p}). \tag{101}$$

Because the distribution is isotropic, the momentum is equally distributed among the three spatial directions, so that

$$p_1^2 = p_2^2 = p_3^2 = \frac{1}{3}|\mathbf{p}|^2. \tag{102}$$

Therefore,

$$p = \frac{1}{3} \int \frac{d^3p}{E}\, |\mathbf{p}|^2 f(\mathbf{p}). \tag{103}$$

Since radiation is relativistic, $E = |\mathbf{p}|$, and thus

$$p = \frac{1}{3} \int d^3p\, E\, f(\mathbf{p}) = \frac{1}{3}\rho. \tag{104}$$

Hence,

$$\boxed{w_{\text{rad}} = \frac{p}{\rho} = \frac{1}{3}.} \tag{105}$$

4.2 Matter: $w = 0$

Matter, in the cosmological sense, refers to nonrelativistic matter, often idealized as pressureless dust. By definition, this component consists of particles whose momenta are much smaller than their masses:

$$|\mathbf{p}| \ll m, \qquad E \simeq m + \frac{|\mathbf{p}|^2}{2m}. \tag{106}$$

The energy density is then dominated by the rest-mass contribution,

$$\rho \simeq \int d^3p\, m\, f(\mathbf{p}), \tag{107}$$

while the pressure is

$$p = \frac{1}{3} \int \frac{d^3p}{E}\, |\mathbf{p}|^2 f(\mathbf{p}) \simeq \frac{1}{3} \int \frac{d^3p}{m}\, |\mathbf{p}|^2 f(\mathbf{p}). \tag{108}$$

Since $|\mathbf{p}|^2/m \ll m$ in the nonrelativistic limit, the pressure is negligible compared with the energy density:

$$p \ll \rho. \tag{109}$$

Thus, in cosmology, nonrelativistic matter is idealized as pressureless:

$$\boxed{p_{\text{m}} = 0.} \tag{110}$$

Substituting this into the perfect-fluid stress–energy tensor yields

$$T_{\mu\nu}^{(\text{m})} = \rho\, u_\mu u_\nu, \tag{111}$$

which is the standard form of dust. Therefore,

$$\boxed{w_{\text{m}} = 0.} \tag{112}$$

4.3 Cosmological Constant: $w = -1$

The cosmological constant may be interpreted as a vacuum energy component. In Einstein's field equation,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{113}$$

the cosmological constant term can be moved to the right-hand side and interpreted as an effective stress–energy tensor:

$$T^{(\Lambda)}_{\mu\nu} = -\frac{\Lambda}{8\pi G} g_{\mu\nu}. \tag{114}$$

Defining the vacuum energy density by

$$\boxed{\rho_\Lambda \equiv \frac{\Lambda}{8\pi G},} \tag{115}$$

this becomes

$$\boxed{T^{(\Lambda)}_{\mu\nu} = -\rho_\Lambda g_{\mu\nu}.} \tag{116}$$

To compare this with the perfect-fluid form,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \tag{117}$$

we observe that $T^{(\Lambda)}_{\mu\nu}$ is proportional only to $g_{\mu\nu}$. Therefore, the coefficient of $u_\mu u_\nu$ must vanish:

$$\rho_\Lambda + p_\Lambda = 0. \tag{118}$$

Hence,

$$p_\Lambda = -\rho_\Lambda. \tag{119}$$

Therefore,

$$\boxed{w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1.} \tag{120}$$

4.4 Summary

Thus, starting from the definition of the stress–energy tensor and imposing the cosmological definitions of the three dominant energy components, one obtains:

$$\boxed{w_{\text{rad}} = \frac{1}{3}, \qquad w_{\text{m}} = 0, \qquad w_\Lambda = -1.} \tag{121}$$

These values have clear physical origins. Radiation is a relativistic and isotropic momentum distribution, which implies $p = \rho/3$. Matter is nonrelativistic and pressureless on cosmological scales, which implies $p = 0$. The cosmological constant is defined by a stress–energy tensor proportional to the metric, which implies $p = -\rho$. These three equation-of-state parameters determine how each component evolves as the universe expands.

5 Scaling with the Scale Factor of each Energy components

In cosmology, the three dominant idealized energy components of the universe are radiation, nonrelativistic matter, and the cosmological constant. Their evolution with the scale factor follows from the energy–momentum conservation law in an FLRW spacetime,

$$\nabla_\mu T^{\mu\nu} = 0. \tag{122}$$

For a homogeneous and isotropic perfect fluid, this conservation law reduces to the continuity equation

$$\boxed{\dot{\rho} + 3H(\rho + p) = 0,} \tag{123}$$

where

$$H \equiv \frac{\dot{a}}{a} \tag{124}$$

is the Hubble parameter.

Using the equation of state

$$p = w\rho, \tag{125}$$

the continuity equation becomes

$$\dot{\rho} + 3H\rho(1 + w) = 0. \tag{126}$$

Substituting $H = \dot{a}/a$, one obtains

$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho(1 + w) = 0. \tag{127}$$

Dividing by ρ and rewriting in differential form,

$$\frac{\dot{\rho}}{\rho} = -3(1 + w)\frac{\dot{a}}{a} \qquad \Longrightarrow \qquad \frac{d\rho}{\rho} = -3(1 + w)\frac{da}{a}. \tag{128}$$

Integrating both sides gives

$$\ln \rho = -3(1 + w) \ln a + \text{constant}, \tag{129}$$

and therefore

$$\boxed{\rho(a) \propto a^{-3(1+w)}.} \tag{130}$$

5.1 Radiation

Radiation is characterized by the equation-of-state parameter

$$w_r = \frac{1}{3}. \tag{131}$$

Substituting this into the general scaling law yields

$$\rho_r(a) \propto a^{-3\left(1+\frac{1}{3}\right)} = a^{-4}. \tag{132}$$

Hence,

$$\boxed{\rho_r \propto a^{-4}.} \tag{133}$$

This result can also be understood physically. The number density of relativistic particles decreases as a^{-3} because of the expansion of volume, while the energy of each particle redshifts as a^{-1}. Therefore the total energy density scales as

$$\rho_r \propto a^{-3}a^{-1} = a^{-4}. \tag{134}$$

5.2 Matter

Nonrelativistic matter is characterized by

$$w_m = 0. \tag{135}$$

Therefore,

$$\rho_m(a) \propto a^{-3(1+0)} = a^{-3}. \tag{136}$$

Hence,

$$\boxed{\rho_m \propto a^{-3}.} \tag{137}$$

Physically, the energy density of matter decreases only because the number density is diluted by the expansion of the universe. Since the energy of each nonrelativistic particle is dominated by its rest mass, there is no additional redshift factor analogous to the radiation case.

5.3 Cosmological Constant

The cosmological constant corresponds to

$$w_\Lambda = -1. \tag{138}$$

Substituting this into the general scaling law gives

$$\rho_\Lambda(a) \propto a^{-3(1-1)} = a^0. \tag{139}$$

Thus,

$$\boxed{\rho_\Lambda = \text{constant.}} \tag{140}$$

Equivalently, since $p_\Lambda = -\rho_\Lambda$, the combination $\rho + p$ vanishes in the continuity equation, and hence

$$\dot{\rho}_\Lambda = 0. \tag{141}$$

5.4 Relation Between the Scale Factor and Time

The evolution of the scale factor with cosmic time is determined by the Friedmann equation. For a spatially flat universe $(k = 0)$ dominated by a single energy component, the first Friedmann equation is

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho.} \tag{142}$$

Using

$$\rho(a) \propto a^{-3(1+w)}, \tag{143}$$

one obtains

$$\left(\frac{\dot{a}}{a}\right)^2 \propto a^{-3(1+w)}. \tag{144}$$

Taking the square root,

$$\frac{\dot{a}}{a} \propto a^{-\frac{3}{2}(1+w)}. \tag{145}$$

Therefore,

$$\dot{a} \propto a^{1-\frac{3}{2}(1+w)}. \tag{146}$$

Solving this differential equation yields, for $w \neq -1$,

$$\boxed{a(t) \propto t^{\frac{2}{3(1+w)}}.} \tag{147}$$

5.4.1 Radiation-Dominated Era

For radiation, $w = \frac{1}{3}$, and therefore

$$a(t) \propto t^{\frac{2}{3\left(1+\frac{1}{3}\right)}} = t^{1/2}. \tag{148}$$

Hence,

$$\boxed{a(t) \propto t^{1/2} \qquad \text{(radiation domination).}} \tag{149}$$

5.4.2 Matter-Dominated Era

For nonrelativistic matter, $w = 0$, and therefore

$$a(t) \propto t^{\frac{2}{3(1+0)}} = t^{2/3}. \tag{150}$$

Hence,

$$\boxed{a(t) \propto t^{2/3} \qquad \text{(matter domination).}} \tag{151}$$

5.4.3 Cosmological-Constant-Dominated Era

For the cosmological constant, $w = -1$, so the power-law formula no longer applies. Since ρ_Λ is constant, the Friedmann equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_\Lambda = \text{constant.} \tag{152}$$

Thus,

$$\frac{\dot{a}}{a} = H_\Lambda, \tag{153}$$

where H_Λ is a constant. Integrating,

$$\frac{da}{a} = H_\Lambda \, dt \qquad \Longrightarrow \qquad \ln a = H_\Lambda t + \text{constant.} \tag{154}$$

Therefore,

$$\boxed{a(t) \propto e^{H_\Lambda t} \qquad \text{(Λ-domination).}} \tag{155}$$

Using $\rho_\Lambda = \Lambda/(8\pi G)$, this may also be written as

$$H_\Lambda = \sqrt{\frac{\Lambda}{3}}, \tag{156}$$

so that

$$a(t) \propto e^{\sqrt{\Lambda/3}\, t}. \tag{157}$$

5.5 Summary

The three dominant cosmological energy components obey the following scaling relations:

$$\boxed{\rho_r \propto a^{-4}, \qquad \rho_m \propto a^{-3}, \qquad \rho_\Lambda = \text{constant.}} \tag{158}$$

When each component dominates the cosmic expansion, the scale factor evolves as

$$\boxed{a(t) \propto t^{1/2} \quad \text{(radiation domination),}} \tag{159}$$

$$\boxed{a(t) \propto t^{2/3} \quad \text{(matter domination),}} \tag{160}$$

$$\boxed{a(t) \propto e^{H_\Lambda t} \quad \text{(Λ-domination).}} \tag{161}$$

These relations form the basis of the standard cosmological description of the thermal history of the universe.

6 Spatial Curvature of the Universe

In FLRW cosmology, the large-scale universe is assumed to be homogeneous and isotropic. Under these symmetry conditions, the spacetime metric takes the form

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \right], \tag{162}$$

where k is the spatial curvature parameter. The parameter k determines the geometry of the spatial hypersurfaces at fixed cosmic time. In particular, it specifies whether the three-dimensional space is positively curved, flat, or negatively curved.

The physically relevant values of k are conventionally normalized to

$$k = +1, \qquad k = 0, \qquad k = -1. \tag{163}$$

These three cases correspond respectively to a closed universe, a flat universe, and an open universe.

6.1 Closed Universe: $k = +1$

When

$$k = +1, \tag{164}$$

the spatial hypersurfaces have positive curvature. This case is called a *closed universe*. A useful two-dimensional analogy is the surface of a sphere. Although every point and every direction are equivalent, the space is not flat; rather, it is curved in such a way that the geometry differs from ordinary Euclidean space.

In a positively curved space, the sum of the angles of a triangle is greater than 180°, and the circumference of a circle is smaller than the Euclidean value $2\pi r$. In this sense, the geometry is “closed” and sphere-like.

Using the curvature density parameter

$$\Omega_k \equiv -\frac{k}{a^2 H^2}, \tag{165}$$

one finds that for $k = +1$,

$$\Omega_k < 0. \tag{166}$$

Equivalently, if

$$\Omega_{\text{tot}} \equiv \Omega_r + \Omega_m + \Omega_\Lambda, \tag{167}$$

then the Friedmann equation implies

$$1 = \Omega_{\text{tot}} + \Omega_k. \tag{168}$$

Thus, a closed universe corresponds to

$$\Omega_{\text{tot}} > 1. \tag{169}$$

6.2 Flat Universe: $k = 0$

When

$$k = 0, \tag{170}$$

the spatial hypersurfaces have zero curvature. This is called a *flat universe*. In this case, the spatial geometry is Euclidean, and the familiar rules of flat-space geometry apply: the sum of the angles of a triangle is 180°, parallel lines remain parallel, and the circumference of a circle is $2\pi r$.

For a flat universe,

$$\Omega_k = 0, \tag{171}$$

and therefore

$$\Omega_{\text{tot}} = 1. \tag{172}$$

This means that the total energy density of the universe is exactly equal to the critical density.

6.3 Open Universe: $k = -1$

When

$$k = -1, \tag{173}$$

the spatial hypersurfaces have negative curvature. This is called an *open universe.* A useful two-dimensional analogy is a saddle-shaped surface. Such a space is homogeneous and isotropic, yet its geometry is hyperbolic rather than Euclidean.

In a negatively curved space, the sum of the angles of a triangle is less than 180°, and the circumference of a circle is larger than the Euclidean value $2\pi r$. Thus, the geometry is more "open" than flat space.

For $k = -1$,

$$\Omega_k > 0, \tag{174}$$

and therefore

$$\Omega_{\text{tot}} < 1. \tag{175}$$

Hence, an open universe corresponds to a total energy density smaller than the critical density.

6.4 Curvature and the Friedmann Equation

The first Friedmann equation may be written as

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \tag{176}$$

Dividing by H^2, one obtains

$$1 = \Omega_r + \Omega_m + \Omega_\Lambda + \Omega_k, \tag{177}$$

where

$$\Omega_r = \frac{\rho_r}{\rho_c}, \qquad \Omega_m = \frac{\rho_m}{\rho_c}, \qquad \Omega_\Lambda = \frac{\Lambda}{3H^2}, \qquad \Omega_k = -\frac{k}{a^2 H^2}, \tag{178}$$

and

$$\rho_c = \frac{3H^2}{8\pi G} \tag{179}$$

is the critical density.

Strictly speaking, Ω_k is not an independent material energy component in the same sense as radiation, matter, or vacuum energy. Rather, it is a geometric contribution arising from the curvature of spatial hypersurfaces. Nevertheless, in the Friedmann equation it behaves as an additional term, and it is therefore convenient to treat it alongside the other density parameters.

Formally, one may define an effective curvature density ρ_k through

$$\frac{8\pi G}{3}\rho_k \equiv -\frac{k}{a^2}, \tag{180}$$

which implies

$$\rho_k \propto a^{-2}. \tag{181}$$

Comparing this with the general scaling law

$$\rho \propto a^{-3(1+w)}, \tag{182}$$

one may assign the formal equation-of-state parameter

$$w_k = -\frac{1}{3}. \tag{183}$$

However, this should be understood only as a formal analogy, since curvature is not a true fluid component but a property of the spatial geometry itself.

6.5 Geometrical Meaning and Physical Interpretation

It is important to emphasize that the classification into closed, flat, and open universes refers to the geometry of the spatial slices, not directly to the ultimate fate of the universe. In simple matter-dominated cosmological models without a cosmological constant, spatial curvature and cosmic destiny were closely related. However, in modern cosmology, where dark energy or a cosmological constant is included, this simple correspondence no longer holds. A closed universe can still expand forever, and an open universe may have a more complicated dynamical history depending on the relative contributions of the various energy components.

Therefore, the primary meaning of k is geometric:

- $k = +1$: positively curved, closed spatial geometry,
- $k = 0$: flat Euclidean spatial geometry,
- $k = -1$: negatively curved, open spatial geometry.

6.6 Summary

The spatial curvature of the universe in FLRW cosmology is determined by the curvature parameter k, which takes the values $+1$, 0, or -1. These correspond respectively to closed, flat, and open universes. A closed universe has positive spatial curvature and satisfies $\Omega_{\text{tot}} > 1$, a flat universe has zero spatial curvature and satisfies $\Omega_{\text{tot}} = 1$, and an open universe has negative spatial curvature and satisfies $\Omega_{\text{tot}} < 1$.

Although the curvature term may be written in the Friedmann equation in a form analogous to an energy density, it is fundamentally geometric rather than material. Nonetheless, the quantity

$$\Omega_k = -\frac{k}{a^2H^2} \tag{184}$$

provides a convenient way to compare the curvature contribution with the other cosmological density parameters.

7 Hubble's Law, the Age of the Universe, and the Size of the Universe

7.1 Hubble's Law

One of the most important observational discoveries in cosmology is Hubble's law, which states that the recession velocity of a galaxy is proportional to its distance from us. It is written as

$$v = H_0 d, \tag{185}$$

where v is the recession velocity, d is the distance to the galaxy, and H_0 is the present value of the Hubble parameter, called the Hubble constant.

This law can be understood naturally in an expanding FLRW universe. The physical distance to a comoving object is

$$d(t) = a(t)\chi, \tag{186}$$

where $a(t)$ is the scale factor and χ is the comoving coordinate, which remains constant for an object moving only with the Hubble flow. Differentiating with respect to time gives

$$\dot{d}(t) = \dot{a}(t)\chi. \tag{187}$$

Using $d(t) = a(t)\chi$, one finds

$$\dot{d}(t) = \frac{\dot{a}}{a} d(t). \tag{188}$$

Since the Hubble parameter is defined by

$$H(t) \equiv \frac{\dot{a}}{a}, \tag{189}$$

it follows that

$$\dot{d}(t) = H(t)d(t). \tag{190}$$

At the present time $t = t_0$, this becomes

$$v = H_0 d. \tag{191}$$

This is Hubble's law.

7.2 Estimating the Age and Size of the Universe from Hubble's Law

Hubble's law immediately gives a characteristic timescale for the universe,

$$t_H \equiv H_0^{-1}, \tag{192}$$

which is called the Hubble time. Since the dimension of H_0 is inverse time, H_0^{-1} gives the order of magnitude of the age of the universe.

If one makes the naive assumption that the expansion rate has always been constant, then the age of the universe may be roughly estimated as

$$t_0 \sim \frac{1}{H_0}. \tag{193}$$

For example, if

$$H_0 = 68 \text{ km s}^{-1}\,\text{Mpc}^{-1}, \tag{194}$$

then

$$H_0^{-1} \approx 14.4 \text{ Gyr}. \tag{195}$$

This gives only an approximate timescale, because the actual expansion rate changes with time depending on the dominant energy components of the universe.

Hubble's law also provides a characteristic length scale,

$$R_H \equiv \frac{}{H_0}, \tag{196}$$

which is called the Hubble radius or Hubble length. For nearby galaxies, where the redshift is small, one may use

$$v \approx z, \tag{197}$$

so that

$$d \approx \frac{z}{H_0}. \tag{198}$$

Thus, by measuring the redshift of nearby galaxies, one can estimate their distances. More generally, $1/H_0$ sets the characteristic size scale associated with the present cosmic expansion.

However, neither H_0^{-1} nor $1/H_0$ gives the exact age or exact size of the universe. The precise age must be obtained from the Friedmann equation by integrating the full expansion history.

7.3 Flat Universe with Matter and Cosmological Constant

7.3.1 Time Integral

We now calculate the age of the universe by integrating over cosmic time. The age of the universe is

$$T=\int_0^T dt=\int_0^{a(t)}\frac{dt}{da}\,da=\int_0^{a(t)}\frac{da}{\dot a}. \tag{199}$$

Since

$$\frac{\dot a}{a}=H, \tag{200}$$

the Friedmann equation allows us to express $\dot a$ in terms of the scale factor.

The Friedmann equation is

$$H^2=\left(\frac{\dot a}{a}\right)^2=\frac{8\pi G}{3}\rho_{\rm tot}-\frac{k}{a^2}. \tag{201}$$

Multiplying both sides by $(a/a_0)^2$, we obtain

$$\left(\frac{\dot a}{a_0}\right)^2+\frac{k}{a_0^2}=\frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-(1+3\omega)}. \tag{202}$$

Writing this in terms of the density parameter Ω, we have

$$\frac{\dot a}{a_0}=\left[\frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-(1+3\omega)}-\frac{k}{a_0^2}\right]^{1/2}=H_0\left[\Omega_0\left(\frac{a}{a_0}\right)^{-(1+3\omega)}-\Omega_{k,0}\right]^{1/2}. \tag{203}$$

If one uses the relation

$$1=\Omega_0-\Omega_{k,0}, \tag{204}$$

then this becomes

$$\frac{\dot a}{a_0}=H_0\left[\Omega_0\left(\frac{a}{a_0}\right)^{-(1+3\omega)}+1-\Omega_0\right]^{1/2}. \tag{205}$$

Now define

$$x\equiv\frac{a}{a_0}. \tag{206}$$

This quantity can also be expressed in terms of the redshift parameter as

$$x=\frac{1}{1+z}. \tag{207}$$

Substituting this into the time integral, we obtain

$$T=\frac{1}{H_0}\int_0^{(1+z)^{-1}}\frac{dx}{\left[1-\Omega_0+\Omega_0x^{-(1+3\omega)}\right]^{1/2}}. \tag{208}$$

7.3.2 Time Integral for a Flat Universe Containing Matter and a Cosmological Constant

Now let us consider a flat universe composed of matter and a cosmological constant. In this case, the Friedmann equation becomes

$$\left(\frac{\dot a}{a_0}\right)^2+\frac{k}{a_0^2}=\frac{8\pi G}{3}\left[\rho_{m,0}\left(\frac{a}{a_0}\right)^{-(1+3\omega_m)}+\rho_{\Lambda,0}\left(\frac{a}{a_0}\right)^{-(1+3\omega_\Lambda)}\right]. \tag{209}$$

Since

$$\omega_m=0,\qquad \omega_\Lambda=-1, \tag{210}$$

the time integral becomes

$$T = \frac{1}{H_0}\int_0^{(1+z)^{-1}} \frac{dx}{\left[\Omega_{m,0}/x + \Omega_{\Lambda,0}x^2\right]^{1/2}}. \tag{211}$$

For convenience, let

$$\Omega \equiv \Omega_{\Lambda,0}. \tag{212}$$

Since the universe is flat,

$$\Omega_{m,0} = 1 - \Omega_{\Lambda,0} = 1 - \Omega. \tag{213}$$

Thus the integral becomes

$$T = \frac{1}{H_0}\int_0^{(1+z)^{-1}} \frac{dx}{\left[(1-\Omega)/x + \Omega x^2\right]^{1/2}}. \tag{214}$$

Now define

$$1 - \Omega = \epsilon. \tag{215}$$

Then

$$T = \frac{1}{H_0}\int_0^{(1+z)^{-1}} \frac{dx}{\left[\epsilon/x + \Omega x^2\right]^{1/2}}. \tag{216}$$

Multiplying both numerator and denominator by $\sqrt{x}$, the integrand becomes

$$T = \frac{1}{H_0}\int_0^{(1+z)^{-1}} \frac{\sqrt{x}\,dx}{\left[\epsilon + \Omega x^3\right]^{1/2}}. \tag{217}$$

Now let

$$x^{3/2} = u. \tag{218}$$

Then

$$dx = \frac{2}{3\sqrt{x}}\,du. \tag{219}$$

Substituting this, we obtain

$$T = \frac{2}{3H_0\sqrt{\Omega}}\int \frac{du}{\left[\epsilon/\Omega + u^2\right]^{1/2}}. \tag{220}$$

Now define

$$\frac{\epsilon}{\Omega} \equiv a^2. \tag{221}$$

Using the trigonometric substitution

$$u = a\tan\theta, \qquad du = a\sec^2\theta\,d\theta, \tag{222}$$

the integral becomes

$$T = \frac{2}{3H_0\sqrt{\Omega}}\int \frac{a\sec^2\theta\,d\theta}{a\left(1+\tan^2\theta\right)^{1/2}} = \frac{2}{3H_0\sqrt{\Omega}}\int \sec\theta\,d\theta. \tag{223}$$

Since

$$\int \sec\theta\,d\theta = \ln\left|\sec\theta + \tan\theta\right| + C, \tag{224}$$

we may rewrite the result in terms of x as

$$T = \frac{2}{3H_0\sqrt{\Omega}}\left[\ln\left|\frac{\sqrt{a^2+x^3}}{a} + \frac{x^{3/2}}{a}\right|\right]_0^{(1+z)^{-1}}. \tag{225}$$

At the present time, $z = 0$, so the upper limit of integration is 1. Since

$$a^2 = \frac{\epsilon}{\Omega} = \frac{1-\Omega}{\Omega}, \tag{226}$$

we obtain

$$T = \frac{2}{3H_0\sqrt{\Omega}} \ln\left|\frac{1+\sqrt{\Omega}}{(1-\Omega)^{1/2}}\right|. \tag{227}$$

Since $\Omega \equiv \Omega_{\Lambda,0}$, the final result is

$$\boxed{T = \frac{2}{3H_0\sqrt{\Omega_{\Lambda,0}}} \ln\left(\frac{1+\sqrt{\Omega_{\Lambda,0}}}{(1-\Omega_{\Lambda,0})^{1/2}}\right)}. \tag{228}$$

7.3.3 The Age of Our Universe

Finally, let us estimate the age of our universe using observational values. Taking

$$H_0 = 68 \text{ km/s/Mpc}, \qquad \Omega_{\Lambda,0} = 0.69, \tag{229}$$

and using

$$H_0^{-1} \approx 14.4 \text{ Gyr}, \tag{230}$$

we obtain

$$T = \frac{2}{3H_0\sqrt{0.69}} \ln\left(\frac{1+\sqrt{0.69}}{(1-0.69)^{1/2}}\right) \cong \frac{0.956}{H_0}. \tag{231}$$

Therefore,

$$T \cong 0.956 \times 14.4 \text{ Gyr} \cong 13.8 \text{ Gyr}. \tag{232}$$

Hence, the age of our flat universe composed of matter and a cosmological constant is approximately

$$\boxed{T \approx 13.8 \text{ Gyr.}} \tag{233}$$

References

[1] D. Baumann, *Cosmology*, Cambridge University Press, 2022.

[2] V. Mukhanov, *Physical Foundations of Cosmology*, Cambridge University Press, 2005.

[3] E. W. Kolb and M. S. Turner, *The Early Universe*, Addison-Wesley, 1990.