Non-perturbative effects in Quantum Field Theory

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1 Introduction

This is a note for the JCP HEP School in March 2026. In this note, I aim to introduce some non-perturbative effects in QFT.

2 The Witten SU(2) Anomaly

2.1 Ferimonic path integral

Let us consider integration over two Grassmann variables θ_1 and θ_2, which anticommute with each other, $\theta_1\theta_2 = -\theta_2\theta_1$. A general function of these variables can be written as

$$f(\theta_1, \theta_2) = f_0 + f_1\theta_1 + f_2\theta_2 + f_{12}\theta_1\theta_2 \tag{1}$$

where f_0, f_1, f_2 and f_{12} are arbitrary coefficients.

The fundamental properties of Grassmann integration are given as

$$\int d\theta_i = 0 \quad , \quad \int d\theta_i \, \theta_i = 1 \tag{2}$$

For the path integral, we need Gaussian integrals. For two θ_i, we have

$$\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, e^{-\theta_1 A_{12} \theta_2} = \int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \, (1 - A_{12}\theta_1\theta_2) = A_{12} \tag{3}$$

Now say we have n θ_i and n other independent θ_i that we will call $\bar{\theta}_i$. Then consider an integral that is an exponential of something quadratic in them

$$\int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \, e^{-\bar{\theta}_i A_{ij} \theta_j} = \frac{1}{n!} \sum_{\text{permutations}\{i_n, j_n\}} \pm A_{i_1 j_1} \cdots A_{i_n j_n} \tag{4}$$

If we think of A_{ij} as a matrix, this is a sum over all elements $\{i, j\}$ where we choose each row and column once, with the sign from the ordering. The $n!$ for the number of permutations cancels the $1/n!$ in front. So

$$\int \mathrm{d}\bar{\theta}_1 \mathrm{d}\theta_1 \cdots \mathrm{d}\bar{\theta}_n \mathrm{d}\theta_n \, e^{-\bar{\theta}_i A_{ij} \theta_j} = \det(\mathbf{A}) \tag{5}$$

2.2 Topological Structure of SU(2) Gauge Transformations

Suppose we have a gauge field A_μ. Under a gauge transformation $U(x)$, it is mapped to

$$A_\mu^U = U^{-1}(x) A_\mu U(x) - i U^{-1}(x) \partial_\mu U(x) \tag{6}$$

Classically, A_μ and A_μ^U represent the same gauge configuration, and in a gauge-invariant quantum theory they should lead to the same contribution to the path integral.

Gauge transformations $U(x)$ can be classified by their homotopy classes. Since $U(x)$ is a map from spacetime compactified to S^4 into the gauge group SU(2), the relevant topological classification is given by

$$\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2 \tag{7}$$

This means that in four-dimensional Euclidean space, there is a gauge transformation $U(x)$ such that $U(x) \to 1$ as $|x| \to \infty$, and $U(x)$ wraps around the gauge group in such a way that it cannot be continuously deformed to the identity.

2.3 The SU(2) Anomaly

Consider the path integral for a $G = \mathrm{SU}(2)$ gauge theory with a single doublet of a left-handed fermions

$$Z = \int \mathrm{d}\psi\, \mathrm{d}\bar{\psi} \int \mathrm{d}A_\mu\, \exp\left[-\int \mathrm{d}^4x\, \left(\frac{1}{2g^2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}i\not{D}\psi\right)\right] \tag{8}$$

Here the $\not{D} = \gamma^\mu D_\mu$ is the Dirac operator for the SU(2) gauge theory

$$D_\mu = \partial_\mu - igA_\mu^a\frac{\sigma^a}{2} \tag{9}$$

We would like to integrate out the fermions and discuss the effective theory with the fermions eliminated.

As is well known, for a theory with a doublet of Dirac fermions, since eq. (5), we can get

$$\int (\mathrm{d}\psi\mathrm{d}\bar{\psi})_{\mathrm{Dirac}}\, e^{\bar{\psi}iD\psi} = \det\, i\not{D} \tag{10}$$

Here the RHS is the infinite product of all eigenvalues of the hermitian operator $i\not{D}$. Since the Pauli-Villars regularization is available for a Dirac fermion, we can regulate this in a gauge invariant way.

Now, with a doublet of Dirac fermions is exactly the same as two left-handed or Weyl doublets. Hence, the fermion integration with a single Weyl doublet would give the square root of eq. (10),

$$\int (\mathrm{d}\psi\mathrm{d}\bar{\psi})_{\mathrm{Weyl}}\, e^{\bar{\psi}iD\psi} = (\det\, i\not{D})^{1/2} \tag{11}$$

But an ambiguity arises here; the square root has two signs. This leads to trouble.

Picking a particular gauge field A_μ, we are free to define in an arbitrary way the sign of $(\det\, i\not{D})^{1/2}$ for this field. Once this is done, there is no further freedom.

Defined in this way $(\det\, i\not{D})^{1/2}$ is certainly invariant under infinitesimal gauge transformations. But nothing guarantees that $(\det\, i\not{D})^{1/2}$ is invariant under the topologically non-trivial gauge transformation U. Actually, we will see that for any gauge field A_μ,

$$(\det\, i\not{D}(A_\mu))^{1/2} = -(\det\, i\not{D}(A_\mu^U))^{1/2} \tag{12}$$

In other words, if one continuously varies the gauge field from A_μ to A_μ^U, one ends up with the opposite sign of the square root.

Imagine varying the gauge field along a continuous path in field space from A_μ to A_μ^U. For instance, one may consider the gauge field $A_\mu^t = (1-t)A_\mu + tA_\mu^U$, with t varied smoothly from zero to one. Let us follow the flow of the eigenvalues as a function of t. The spectrum of $i\not{D}$ is precisely the same at $t = 1$ as it is at $t = 0$. However, the individual eigenvalues may rearrange themselves as t is varied from zero to one.

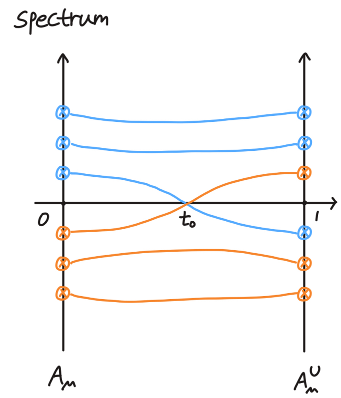

Figure 1: The flow of eigenvalues as the gauge field is varied from A_μ to A_μ^U

In particular, one of the eigenvalues which was positive at $t = 0$ is negative by the time $t = 1$. If at $t = 0$, $(\det i\not{D})^{1/2}$ was defined as the product of the positive eigenvalues, then, following the eigenvalues continuously, by the time we reach $t = 1$ $(\det i\not{D})^{1/2}$ is the product of many positive eigenvalues and a single negative one.

The Atiyah-Singer theorem permits more complicated rearrangements of eigenvalues, but the number of positive eigenvalues that become negative as t is varied from 0 to 1 is always odd.

2.4 Mod 2 index for 5d Dirac operator

Consider a five-dimensional cylinder $S^4 \times \mathbb{R}$. Let x^μ $(\mu = 1, 2, 3, 4)$ be coordinates for S^4 while the position in the time direction is called τ. Here, we can consider a gauge connection $\mathcal{A}$ on the 5d space with $\mathcal{A}_\tau = 0$. Meanwhile, for $\mathcal{A}_\mu$ we choose a gauge configuration such that

$$\mathcal{A}_\mu(x, \tau) \to A_\mu(x) \quad \text{as} \quad \tau \to -\infty \tag{13}$$

and

$$\mathcal{A}_\mu(x, \tau) \to A_\mu^U(x) \quad \text{as} \quad \tau \to +\infty \tag{14}$$

Our 5d gauge field $\mathcal{A}(x, \tau)$ smoothly interpolates between a 4d gauge configuration at $\tau \to -\infty$ and a gauge equivalent configuration at $\tau \to +\infty$, related by a non-trivial gauge transformation.

Consider the five-dimensional Dirac operator for an SU(2) doublet of fermions,

$$\not{D}_5 \Psi = \sum_{i=1}^{5} \gamma^i \left(\partial_i + \sum_{a=1}^{3} A_i^a T^a \right) \Psi = \gamma^\tau \frac{\partial \Psi}{\partial \tau} + \not{D}_4 \Psi \tag{15}$$

The spinor Ψ has eight components because the spinor representation of O(5) is four dimensional while an SU(2) doublet has two components.

The spinor representation of O(5) is pseudo-real, rather than real, and the doublet of SU(2) is likewise pseudo-real. But the tensor product of the spinor representation of O(5) with the doublet of SU(2) is a real representation of O(5) × SU(2). This means that in eq. (15), we can take the gamma matrices γ^i to be real, symmetric 8×8 matrices while the anti-hermitian generators T^a of SU(2) are real, anti-symmetric matrices.

The five-dimensional Dirac operator $\not{D}_5$ for an SU(2) doublet is therefore real antisymmetric operator, acting on an infinite dimensional space. The eigenvalues of such a real antisymmetric operator either vanish or are imaginary and occur in complex conjugate pairs.

Any zero mode of the $\not{D}_5$ obeys

$$\frac{\partial \Psi}{\partial \tau} = -\gamma^\tau \not{D}_4 \Psi \tag{16}$$

where $\not{D}_4$ is a 4d Dirac operator at each τ.

Assume that the gauge configuration $\mathcal{A}_\mu(x,\tau)$ evolves adiabatically in τ, then the above eq. (16) can be solved in the adiabatic approximation. This means that the eigenfunction $\Psi(x,\tau)$ can be written as

$$\Psi(x,\tau) = f(\tau)\phi(x) \tag{17}$$

where, for each fixed τ, $\phi(x)$ is an eigenfunction of the 4d Dirac operator

$$\gamma^\tau \not{D}_4 \phi_\tau(x) = \lambda(\tau)\phi_\tau(x) \tag{18}$$

In the adiabatic limit, eq. (16) becomes

$$\frac{\mathrm{d}f}{\mathrm{d}\tau} = -\lambda(\tau) f(\tau) \qquad \Rightarrow \qquad f(\tau) = f_0 \exp\left(-\int_0^\tau \mathrm{d}\tau' \, \lambda(\tau')\right) \tag{19}$$

This is normalizable only if λ is positive for $\tau \to +\infty$, and negative for $\tau \to -\infty$.

Therefore, in the adiabatic approximation, the number of zero eigenvalues of $\not{D}_5$ is equal to the number of eigenvalue curves which pass from negative to positive values between $t = 0$ and $t = 1$.

Meanwhile, the mod 2 Atiyah-Singer index theorem predicts that that the number of zero modes in this 5d gauge field is odd

$$\mathrm{ind}_2\,(i\not{D}_5) = \dim \mathrm{Ker}\,(i\not{D}_5) = 1 \mod 2 \tag{20}$$

Therefore, the number of eigenvalue curves that pass from negative to positive values is odd. It implies that $(\det\, i\not{D})^{1/2}$ is odd under the topologically non-trivial gauge transformation $U(x)$.

An important implication for the SU(2) anomaly is that the number of left-handed Weyl fermion doublets must be even. In the Standard Model, each generation contains one lepton doublet and three quark doublets, so the total number of $\mathrm{SU}(2)_L$ doublets per generation is 4. Hence, the SM is free from the Witten SU(2) anomaly.

3 Instantons and Sphalerons

3.1 Vacuum structure of Yang-Mills gauge theories

SU(2) gauge theory has an infinite number of classical field configurations of zero energy, distinguished by an integer n, and separated by energy barriers. Generically, between two quantum states $|n\rangle$ and $|n'\rangle$ that are separated by an energy barrier, there is a tunneling amplitude of the form

$$\langle n'|H|n\rangle \sim e^{-S} \tag{21}$$

where H is the Hamiltonian, and S is the euclidean action for a classical solution of the euclidean field equations that mediates between the field configuration corresponding to n at $t = -\infty$, and the field configuration corresponding to n' at $t = +\infty$.

For SU(2) gauge theory, there is a classical solution of the euclidean field equations that mediates between states with winding numbers n and n'. The value of this action is $S = |n' - n|S_1$, where $S_1 = 8\pi^2/g^2$, and g is the Yang-Mills coupling constant. For $n' - n = 1$, this solution is the **instanton**.

In next subsection, we will construct the instanton configuration, but first we study the consequences of its existence. For SU(2) gauge theory, eq. (21) reads

$$\langle n'|H|n\rangle \sim e^{-|n'-n|S_1} \tag{22}$$

These matrix elements depend only on $n' - n$, and so H can be diagonlized by **theta vacua** of the form

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta}|n\rangle \tag{23}$$

Also, we can calculate instanton contribution to the path integral. Let us consider the Euclidean path integral, with the boundary condition that we start with a state of winding number n_- at $x_4 = -\infty$, and end with a state with winding number n_+ at $x_4 = +\infty$. The only field configurations that contribute are those with winding number $n_+ - n_-$. We can therefore write

$$Z_{n_+\leftarrow n_-} = \int \mathcal{D}A_{n_+\leftarrow n_-} \exp\left[-\int \mathrm{d}^4x \,\mathrm{Tr}\left(\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)\right] \tag{24}$$

Suppose now that we are interested in starting with a particular theta vacuum $|\theta\rangle$, and ending with a different theta vacuum $|\theta'\rangle$. Then, corresponding path integral is given as

$$Z_{\theta'\leftarrow\theta}(J) = \sum_{n_-,n_+} e^{i(n_+\theta' - in_-\theta)} Z_{n_+\leftarrow n_-}(J) \tag{25}$$

Let $n_+ = n_- + n$, so that $n_+\theta' - n_-\theta = n_-(\theta' - \theta) + n\theta'$,

$$Z_{\theta'\leftarrow\theta} = \sum_{n_-,n} e^{in_-(\theta'-\theta)}e^{in\theta'} \int \mathcal{D}A_{\theta'\leftarrow\theta} \exp\left[-\int \mathrm{d}^4x \,\mathrm{Tr}\left(\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)\right] \tag{26}$$

Summing over n_- then the above equation becomes

$$Z_{\theta'\leftarrow\theta} = \delta(\theta'-\theta)\sum_n e^{in\theta'}\int \mathcal{D}A_n\,\exp\left[-\int \mathrm{d}^4x\,\mathrm{Tr}\left(\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)\right] \tag{27}$$

Finally, we can get

$$Z_\theta := \sum_n e^{in\theta}\int \mathcal{D}A_n\,\exp\left[-\int \mathrm{d}^4x\,\mathrm{Tr}\left(\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} - J^\mu A_\mu\right)\right] \tag{28}$$

Since

$$n = \frac{1}{16\pi^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,(F_{\mu\nu}\tilde{F}^{\mu\nu}) \tag{29}$$

We can write

$$Z_\theta = \int \mathcal{D}A\,\exp\left[\int \mathrm{d}^4x\,\mathrm{Tr}\left(-\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu} + \frac{i\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + J^\mu A_\mu\right)\right] \tag{30}$$

The vacuum angle θ now appears as the coefficient of an extra term in the Yang-Mills Lagrangian.

3.2 Instanton Configuration

For the euclidean Yang-Mills action

$$S_{\mathrm{YM}} = \frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu}) \tag{31}$$

we can construct a Bogomol'nyi bound. Since for euclidean space, we have

$$\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu} \tag{32}$$

It implies that

$$\frac{1}{2}\mathrm{Tr}\,(\tilde{F}_{\mu\nu} \pm F_{\mu\nu})^2 = \mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu}) \pm \mathrm{Tr}\,(\tilde{F}^{\mu\nu}F_{\mu\nu}) \tag{33}$$

Then, the action eq. (31) can be written as

$$S_{\mathrm{YM}} = \frac{1}{2g^2}\int \mathrm{d}^4x\,\left[\frac{1}{2}\mathrm{Tr}\,(\tilde{F}_{\mu\nu} \pm F_{\mu\nu})^2 \mp \mathrm{Tr}\,(\tilde{F}^{\mu\nu}F_{\mu\nu})\right] \tag{34}$$

Also, from the above eq. (33) we can get the fact,

$$\mathrm{Tr}\,(F^{\mu\nu}F_{\mu\nu}) \geq \left|\mathrm{Tr}\,(\tilde{F}^{\mu\nu}F_{\mu\nu})\right| \tag{35}$$

Hence, it gives the bound on action as

$$S_{\mathrm{YM}} = \frac{1}{2g^2}\int \mathrm{d}^4x\,\left[\frac{1}{2}\mathrm{Tr}\,(\tilde{F}_{\mu\nu} \pm F_{\mu\nu})^2 \mp \mathrm{Tr}\,(\tilde{F}^{\mu\nu}F_{\mu\nu})\right] \geq \frac{1}{2g^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,(\tilde{F}^{\mu\nu}F_{\mu\nu}) \tag{36}$$

Therefore, we have

$$S_{\mathrm{YM}} \geq \frac{8\pi^2}{g^2}|n| \tag{37}$$

where n is the winding number,

$$n = \frac{1}{16\pi^2} \int \mathrm{d}^4x \, \mathrm{Tr}\,(\tilde{F}^{\mu\nu} F_{\mu\nu}) \tag{38}$$

Thus, the eq. (37) gives us the minimum value of the euclidean action for a solution of the euclidean field equations that mediates between a vacuum configuration with winding number n_- at $x_4 = -\infty$ and a vacuum configuration with winding number $n_+ = n_- + n$ at $x_4 = +\infty$.

To get an instanton configuration, we have to find the field configuration which satisfies the self dual equation

$$F_{\mu\nu} = \tilde{F}_{\mu\nu} \tag{39}$$

Our starting point is the ansatz

$$A_\mu(x) = \alpha \sigma_{\mu\nu} \partial_\nu \ln f(x) \tag{40}$$

where α is a real constant to be fixed and $\sigma_{\mu\nu}$ is the 2×2 matrix representation of SO(4) algebra as

$$\sigma_{\mu\nu} = \frac{1}{2}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu) \tag{41}$$

and

$$\sigma_\mu = (\vec{\tau}, i\mathbf{1}), \qquad \bar{\sigma}_\mu = (\vec{\tau}, -i\mathbf{1}) \tag{42}$$

where $\vec{\tau}$ are the Pauli matrices.

After some calculation, we can get one-instanton solution as

$$A^a_\mu(x) = \frac{2\eta^a_{\mu\nu}(x - x_0)^\nu}{(x - x_0)^2 + \rho^2} \tag{43}$$

where ρ is the size of instanton and $\eta^a_{\mu\nu}$ is known as the 't Hooft symbol, defined as

$$\eta^a_{\mu\nu} = \begin{cases} \epsilon^{a\mu\nu} & \mu, \nu \in 1, 2, 3 \\ -\delta^{a\nu} & \mu = 0 \\ +\delta^{a\mu} & \nu = 0 \\ 0 & \mu = \nu = 0 \end{cases} \tag{44}$$

means

$$\eta^1_{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \eta^2_{\mu\nu} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \eta^3_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{45}$$

Especially, this one-instanton solution is called by **BPST instanton**.

3.3 Sphaleron Configuration

Consider the $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ electroweak theory

$$\mathcal{L} = -\frac{1}{4}(W^a_{\mu\nu})^2 - \frac{1}{4}B^2_{\mu\nu} + (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi) \tag{46}$$

where $W^a_{\mu\nu}$ are the SU(2) gauge bosons with field strength

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\varepsilon^{abc} W^b_\mu W^c_\nu \tag{47}$$

where ε^{abc} is the totally antisymmetric structure constant, and B_μ is the hypercharge gauge boson with

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{48}$$

The covariant derivative acting on the Higgs doublet is given by

$$D_\mu \Phi = \partial_\mu \Phi - \frac{1}{2} i g W^a_\mu \sigma^a \Phi - \frac{1}{2} i g' B_\mu \Phi \tag{49}$$

where g is the $\mathrm{SU}(2)_W$ coupling, g' is the $\mathrm{U}(1)_Y$ coupling, and σ^a are the Pauli matrices. The Higgs potential takes the standard form

$$V(\Phi) = -\mu^2 |\Phi|^2 + \lambda |\Phi|^4 \tag{50}$$

with mass parameter μ and self-coupling λ.

After spontaneous symmetry breaking, the Higgs field acquires a vacuum expectation value $v = \mu/\sqrt{\lambda} \simeq 246\,\mathrm{GeV}$, giving rise to the W-boson mass $m_W = gv/2$ and the Higgs boson mass $m_H = \sqrt{2\lambda} v$. The electroweak mixing angle is defined as $\tan\theta_W = g'/g$.

Using the Euler-Lagrange equations, we derive the field equations from the Lagrangian eq. (46) as

$$D_\mu D^\mu \Phi = -\frac{\partial V(\Phi)}{\partial \Phi^\dagger} \tag{51}$$

$$D^\nu W^a_{\mu\nu} = -i\frac{f}{2}\left[\Phi^\dagger \sigma^a (D_\mu \Phi) - (D_\mu \Phi)^\dagger \sigma^a \Phi\right] \tag{52}$$

$$\partial^\nu B_{\mu\nu} = -i\frac{g'}{2}\left[\Phi^\dagger (D_\mu \Phi) - (D_\mu \Phi)^\dagger \Phi\right] \tag{53}$$

where we defined the covariant derivative of the field strength as

$$D^\nu W^a_{\mu\nu} = \partial^\nu W^a_{\mu\nu} + g\varepsilon^{abc} W^\nu_b W_{\mu\nu c} \tag{54}$$

We set $g' = 0$ to analyze the sphaleron in the $\theta_W \to 0$ limit, the $\mathrm{U}(1)_Y$ field decouples from the theory. After spontaneous symmetry breaking, the Higgs potential can be rewritten as

$$V(|\Phi|) = \lambda \left(|\Phi|^2 - \frac{v^2}{2}\right)^2 \tag{55}$$

which has a minimum at $|\Phi| = v/\sqrt{2}$. The field equations (51) - (53) simplify to

$$D_\mu D^\mu \Phi = -2\lambda \left(|\Phi|^2 - \frac{v^2}{2}\right) \Phi \tag{56}$$

$$D^\nu W^a_{\mu\nu} = -i\frac{g}{2}\left[\Phi^\dagger \sigma^a (D_\mu \Phi) - (D_\mu \Phi)^\dagger \sigma^a \Phi\right] \tag{57}$$

To capture the essential features of the sphaleron configuration, we introduce a general spherically symmetric ansatz as

$$W_0^a(x) = \frac{1}{g} G(r,t) \frac{x_a}{r} \tag{58}$$

$$W_j^a(x) = \frac{1}{g} \left[\frac{(f_A(r,t)-1)}{r^2} \varepsilon_{jam} x_m + \frac{f_B(r,t)}{r^3} (r^2 \delta_{ja} - x_j x_a) + \frac{f_C(r,t)}{r^2} x_j x_a \right] \tag{59}$$

$$\Phi(x) = \frac{v}{\sqrt{2}} \left[H(r,t) + i K(r,t) \frac{\boldsymbol{\sigma} \cdot \mathbf{x}}{r} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{60}$$

where $r = |\mathbf{x}|$ is the radial coordinate and $\boldsymbol{\sigma} \cdot \mathbf{x} = \sigma^a x_a$.

Substituting the ansatz eq. (58) - (60) into the field equations (51) - (53), we can get a system of six coupled partial differential equations for the functions $G(r,t)$, $f_A(r,t)$, $f_B(r,t)$, $f_C(r,t)$, $H(r,t)$, and $K(r,t)$.

For static configurations, all time derivatives vanish and the time component of the gauge field must be zero. It implies that

$$0 = W_0^a(x) = \frac{1}{g} G(r,t) \frac{x_a}{r} \qquad \Rightarrow \qquad G(r,t) = 0 \tag{61}$$

Additionally, we adopt the radial gauge condition $x^i W_i^a = 0$. This implies that

$$\begin{aligned} 0 = x^j W_j^a(x) = \frac{1}{g} \left[\frac{(f_A(r,t)-1)}{r^2} x^j \varepsilon_{jam} x_m + \frac{f_B(r,t)}{r^3} (r^2 x^j \delta_{ja} - x^j x_j x_a) + \frac{f_C(r,t)}{r^2} x^j x_j x_a \right] \\ \Rightarrow \qquad f_C(r,t) = 0 \end{aligned} \tag{62}$$

In the radial gauge, the gauge potential has no radial component. This property implies that at large radius $r \to \infty$, a finite-energy field configuration must asymptotically approach a pure gauge. Therefore, we can solve the SU(2)$_W$ equations of motion as $r \to \infty$ by using

$$\sigma^a W_j^a(\infty) = -\frac{2i}{g} \partial_j U U^{-1}, \qquad \Phi(\infty) = \frac{v}{\sqrt{2}} U \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{63}$$

where the SU(2)$_W$ group element $U(r)$ is given by

$$U(r) = \exp\left(\frac{i}{2} q \frac{\boldsymbol{\sigma} \cdot \mathbf{r}}{r} \right) \tag{64}$$

Here $q \in [0, 4\pi]$ is a free parameter. The pure gauge condition also implies that $W_{ij}^a(r \to \infty) = 0$. Physically, this can be understood as a consequence of the finite energy constraint, which forces the fields to approach a pure gauge + VEV configuration at spatial infinity.

3.4 Chern-Simons number of the Sphaleron configuration

In the Standard Model, the baryon number current J_B^μ is not conserved at the quantum level due to the chiral anomaly. Instead, it satisfies an anomaly equation

$$\partial_\mu J_B^\mu = \frac{N_f g^2}{32\pi^2} \mathrm{Tr}\left(W_{\mu\nu} \widetilde{W}^{\mu\nu} \right) \tag{65}$$

where $N_f = 3$ is the number of fermion families, $\widetilde{W}^{a\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} W_{\rho\sigma}^a$ is the field dual.

The RHS of eq. (65) can be rewritten as the four-divergence of a current K^μ, known as the **Chern-Simons current**

$$\frac{g^2}{32\pi^2}\mathrm{Tr}\,(W_{\mu\nu}\widetilde{W}^{\mu\nu}) = \partial_\mu K^\mu \tag{66}$$

where K^μ is given by

$$K^\mu = \frac{g^2}{16\pi^2}\varepsilon^{\mu\nu\rho\sigma}(W_\nu^a\partial_\rho W_\sigma^a - \frac{g}{3}\varepsilon^{abc}W_\nu^a W_\rho^b W_\sigma^c) \tag{67}$$

When instanton solution exist in Euclidean spacetime, the integral of the anomaly over all spacetime gives a topological winding number

$$N = \frac{g^2}{32\pi^2}\int \mathrm{d}^4x\,\mathrm{Tr}\,(W_{\mu\nu}\widetilde{W}^{\mu\nu}) \tag{68}$$

where N takes only integer values because of the topological structure of the winding number. This from the fact that we can rewrite the above equation as

$$n = \frac{1}{8\pi^2}\int \mathrm{Tr}\,(W\wedge W) = C_2 \tag{69}$$

where C_2 is the second Chern number of the SU(2) bundle, which is the topological invariant.

Consider the situation that starting at the trivial vacuum when $t=-\infty$ and arriving at the sphaleron configuration when $t=t_0$. Using Stoke's theorem, we can express the topological index in terms of the time component of the Chern-Simons current

$$N(t_0) = \int \mathrm{d}^3x\,K^0\Big|_{t=t_0}^{t=-\infty} + \int_{-\infty}^{t_0}\mathrm{d}t\int_S \mathbf{K}\cdot\mathrm{d}\mathbf{S} \tag{70}$$

where S is the surface at spatial infinite. If $\mathbf{K}$ decreases sufficiently rapidly as $r\to\infty$, the surface integral vanishes, and we can define the Chern-Simons number at time t_0 as

$$N_{\mathrm{CS}}(t_0) = \int \mathrm{d}^3x\,K^0(t_0) \tag{71}$$

assuming $K^0=0$ at $t=-\infty$ for simplicity.

In the radial gauge we have adopted, the topological number N receives an additional contribution from a non-vanishing surface term, leading to

$$N = \int \mathrm{d}^3x\,K^0 + \frac{q-\sin q}{2\pi} \tag{72}$$

$$= \frac{1}{2\pi}\left(\int_{r=0}^{r=\infty}\mathrm{d}r\,(f_A'f_B - f_Af_B') + f_B\Big|_{r=0}^{r=\infty}\right) + \frac{q-\sin q}{2\pi} \tag{73}$$

where q is the parameter in the pure gauge configuration. In Manton parametrization, we can set $f_B(r,t)=0$ and it leads the first term of the above equation vanish.

A vacuum of the $\mathrm{SU}(2)_W$ theory has the Higgs field at its VEV magnitude, $|\Phi| = v/\sqrt{2}$, and the gauge field in a pure gauge configuration. Such configurations are characterized by integer Chern-Simons numbers, $N_{\mathrm{CS}} = n\in\mathbb{Z}$. These different vacua, labeled by consecutive integers, are separated by energy barriers in the configuration space. The sphaleron represents the static solution at the top of this barrier. By symmetry considerations, the sphaleron between vacua with $N_{\mathrm{CS}} = n$ and $N_{\mathrm{CS}} = n + 1$ has $\mathrm{N}_{\mathrm{CS}} = n + \frac{1}{2}$.

$$N_{\mathrm{CS}} = n + \frac{1}{2} \tag{74}$$

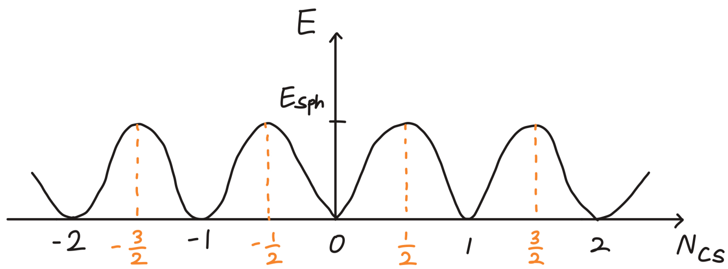

Figure 2: Distinguished vacua with CS number

4 Pole structure

4.1 Källén-Lehmann representation

Consider the two-point function without time-ordering

$$\langle \Omega | \phi(x) \phi(y) | \Omega \rangle \tag{75}$$

The general fields $\phi(x)$ are Heisenberg picture operator acting on the Hilbert space. It means

$$e^{-i\hat{P}x} \phi(x) e^{i\hat{P}x} = \phi(0) \tag{76}$$

If we have a state $|X\rangle$ with momentum p^μ, so $\hat{P}^\mu |X\rangle = p_X^\mu |X\rangle$, then we have

$$\begin{aligned} \langle \Omega | \phi(x) | X \rangle &= \langle \Omega | e^{i\hat{P}x} e^{-i\hat{P}x} \phi(x) e^{i\hat{P}x} e^{-i\hat{P}x} | X \rangle & (77) \\ &= e^{-ip_X x} \langle \Omega | \phi(x) | X \rangle & (78) \end{aligned}$$

where $\langle \Omega | \hat{P} = 0$ has been used.

Insert the complete set

$$1 = \sum_X \int \mathrm{d}\Pi_X |X\,\rangle \langle X| \tag{79}$$

where the sum is over single- and multi-particle states $|X\rangle$ and

$$\mathrm{d}\Pi_X := \prod_{j \in X} \frac{\mathrm{d}^3 p_j}{(2\pi)^3} \frac{1}{2E_j} \tag{80}$$

to the eq. (75) as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \sum_X \int \mathrm{d}\Pi_X \, e^{-ip_X(x-y)}|\langle\Omega|\phi(0)|X\rangle|^2 \tag{81}$$

$$= \int \frac{\mathrm{d}^4p}{(2\pi)^4} e^{-ip(x-y)} \left[\sum_X \int \mathrm{d}\Pi_X \, (2\pi)^4\delta^{(4)}(p-p_X)|\langle\Omega|\phi(0)|X\rangle|^2\right] \tag{82}$$

Here the quantity in brackets of the above equation is a Lorentz scalar, so it can only depend on p^2. Since the states $|X\rangle$ are physical, on-shell states in the Hilbert space, they all have momentum p_X^μ with $p_X^2 \geq 0$ and positive energy. Thus $p^2 \geq 0$ and $p^0 > 0$ as well. Therefore, we can write

$$\sum_X \int \mathrm{d}\Pi_X \, (2\pi)^4\delta^{(4)}(p-p_X)|\langle\Omega|\phi(0)|X\rangle|^2 = 2\pi\theta(p^0)\rho(p^2) \tag{83}$$

where $\rho(p^2)$ is known as a **spectral density**.

Finally, we can write the two-point function eq. (75) as

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^3} e^{-ip(x-y)}\theta(p^0)\rho(p^2) \tag{84}$$

To simplify this further we define

$$D(x,y,m^2) := \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_p} e^{-ip(x-y)} \tag{85}$$

$$= \int \frac{\mathrm{d}^4p}{(2\pi)^3} \, e^{-ip(x-y)}\theta(p_0)\delta(p^2-m^2) \tag{86}$$

where $\omega_p = \sqrt{\vec{p}^2+m^2}$. This leads

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \int_0^\infty \mathrm{d}q^2 \, \rho(q^2)D(x,y,q^2) \tag{87}$$

To connect to S-matrix elements, we need to relate the spectral function to time-ordered products as

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \langle\Omega|\phi(x)\phi(y)|\Omega\rangle\theta(x^0-y^0) + \langle\Omega|\phi(y)\phi(x)|\Omega\rangle\theta(y^0-x^0) \tag{88}$$

$$= \int_0^\infty \mathrm{d}q^2 \, \rho(q^2)\left[D(x,y,q^2)\theta(x^0-y^0) + D(y,z,q^2)\theta(y^0-x^0)\right] \tag{89}$$

Since the identity

$$D(x,y,q^2)\theta(x^0-y^0) + D(y,x,q^2)\theta(y^0-x^0) = \int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{i}{p^2-q^2+i\varepsilon}e^{ip(x-y)} \tag{90}$$

We can get

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \int \frac{\mathrm{d}^4p}{(2\pi)^4}e^{ip(x-y)}i\Pi(p^2) \tag{91}$$

where

$$\Pi(p^2) := \int_0^\infty \mathrm{d}q^2 \frac{\rho(q^2)}{p^2-q^2+i\varepsilon} \tag{92}$$

is known as the **spectral representation** or **Källén-Lehmann representation**.

The spectral density has a lot of information in it. Basically it tells us about all the on-shell intermidiate states in the theory. It is observable (in principle) since it is just based on an observable Green's function, $\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$. For a free theory

$$\Pi(p^2) = \frac{1}{p^2 - m^2 + i\varepsilon} \tag{93}$$

and $\rho(q^2) = \delta(q^2 - m^2)$. For an interacting theory, the spectral function will have singularities at locations of physical, renormalized particle masses and other physical thresholds. Since $\rho(q^2)$ is real and $\rho(q^2) > 0$, we can calculate if from the 2-point function by taking the imaginary part of $\Pi(q^2)$,

$$\rho(p^2) = -\frac{1}{\pi}\mathrm{Im}\left[\Pi(p^2)\right] \tag{94}$$

In a unitary theory $\Pi(p^2)$ can have an imaginary part only when cuts can put intermediate particles on-shell. Thus, the spectral density contains information about the particles in the theory. In particular, it can tell us about these particles regardless of whether there are fundamental fields corresponding to them in the Lagrangian.

4.2 Polology

What we want to look at now is the n-point green function

$$G_n(p_1, \cdots, p_n) := \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \, e^{ip_1\cdot x_1} \cdots e^{ip_n\cdot x_n} \, \langle\Omega|\phi(x_1)\cdots\phi(x_n)|\Omega\rangle \tag{95}$$

Suppose we choose a specific subset of momenta going

$$q := p_1 + p_2 + \cdots + p_{r-1} + p_r \tag{96}$$

$$= -p_{r+1} - p_{r+2} - \cdots - p_{n-1} - p_n \tag{97}$$

where $1 \le r \le n-1$.

Let split the green function eq. (95) as

$$\begin{aligned} G_n(p_1, \cdots, p_n) = \int \mathrm{d}^4x_1 \cdots \mathrm{d}^4x_n \, e^{ip_1\cdot x_1} \cdots e^{ip_n\cdot x_n} \\ \times \, \langle\Omega|T\{\phi(x_1)\cdots\phi(x_r)\}T\{\phi(x_{r+1})\cdots\phi(x_n)\}|\Omega\rangle \, + \text{extra} \end{aligned} \tag{98}$$

Inserting a complete set of states implies as

$$\begin{aligned} &\langle\Omega|T\{\phi(x_1)\cdots\phi(x_r)\}T\{\phi(x_{r+1})\cdots\phi(x_n)\}|\Omega\rangle \, + \text{extra} \\ &= \int \frac{\mathrm{d}^3p_\Psi}{(2\pi)^3}\frac{1}{2E_\Psi}\langle\Omega|T\{\phi(x_1)\cdots\phi(x_r)\}|\Psi\rangle\langle\Psi|T\{\phi(x_{r+1})\cdots\phi(x_n)\}|\Omega\rangle \, + \text{extra} \end{aligned} \tag{99}$$

Here I introduced the one-particle state $|\Psi\rangle$ of mass m.

Then, the $\langle\Omega|T\{\phi(x_1)\cdots\phi(x_r)\}|\Psi\rangle$ term becomes

$$\begin{aligned}
&\langle\Omega|T\{\phi(x_1)\cdots\phi(x_r)\}|\Psi\rangle &(100)\\
&\qquad= \langle\Omega|T\{e^{i\hat{P}x_1}e^{-i\hat{P}x_1}\phi(x_1)e^{i\hat{P}x_1}e^{-i\hat{P}x_1}\phi(x_2)e^{i\hat{P}x_1}\cdots e^{-i\hat{P}x_1}\phi(x_r)e^{i\hat{P}x_1}e^{-i\hat{P}x_1}\}|\Psi\rangle &(101)\\
&\qquad= e^{-ip_\Psi x_1}\langle\Omega|T\{\phi(0)\phi(x_2-x_1)\cdots\phi(x_r-x_1)\}|\Psi\rangle &(102)\\
&\qquad= e^{-ip_\Psi x_1}\langle\Omega|T\{\phi(0)\phi(y_2)\cdots\phi(y_r)\}|\Psi\rangle &(103)
\end{aligned}$$

where we have defined $y_i = x_i - x_1$ with $2 \le i \le r$.

Similarly we can do same process for $\langle\Psi|T\{\phi(x_{r+1})\cdots\phi(x_n)\}|\Omega\rangle$,

$$\begin{aligned}
&\langle\Psi|T\{\phi(x_{r+1})\cdots\phi(x_n)|\Omega\rangle &(104)\\
&= \langle\Psi|e^{i\hat{P}x_{r+1}}e^{-i\hat{P}x_{r+1}}\phi(x_{r+1})e^{i\hat{P}x_{r+1}}e^{-i\hat{P}x_{r+1}}\phi(x_{r+2})e^{i\hat{P}x_{r+1}}\cdots e^{-i\hat{P}x_{r+1}}\phi(x_n)e^{i\hat{P}x_{r+1}}e^{-i\hat{P}x_{r+1}}|\Omega\rangle &(105)\\
&= e^{ip_\Psi x_{r+1}}\langle\Psi|\phi(0)\phi(x_{r+2}-x_{r+1})\cdots\phi(x_n-x_{r+1})|\Omega\rangle &(106)\\
&= e^{ip_\Psi x_{r+1}}\langle\Psi|\phi(0)\phi(y_2)\cdots\phi(y_n)|\Omega\rangle &(107)
\end{aligned}$$

where we have defined $y_j = x_j - x_{r+1}$ with $r+2 \le j \le n$.

Therefore, changing variables on all except x_1 and x_{r+1}, we have

$$\int \mathrm{d}^4x_1\cdots\mathrm{d}^4x_n = \int \mathrm{d}^4x_1\mathrm{d}^4x_{r+1}\int \mathrm{d}^4y_2\cdots\mathrm{d}^4y_r\mathrm{d}^4y_{r+2}\cdots\mathrm{d}^4y_n \tag{108}$$

and

$$\begin{aligned}
e^{ip_1x_1}\cdots e^{ip_nx_n} &= e^{i(p_1x_1+p_2x_2+\cdots+p_rx_r)}\cdot e^{i(p_{r+1}x_{r+1}+p_{r+2}x_{r+2}+\cdots+p_nx_n)} &(109)\\
&= e^{i(p_1x_1+p_2(y_2+x_1)+\cdots+p_r(y_r+x_1))}\cdot e^{i(p_{r+1}x_{r+1}+p_{r+2}(y_{r+2}+x_{r+1})+\cdots+p_n(y_n+x_{r+1}))} &(110)\\
&= e^{i(p_1+p_2+\cdots+p_r)\cdot x_1}\cdot e^{i(p_2y_2+\cdots+p_ry_r)}\cdot e^{i(p_{r+1}+p_{r+2}+\cdots+p_n)\cdot x_{r+1}}\cdot e^{i(p_{r+2}x_{r+2}+\cdots+p_ny_n)} &(111)
\end{aligned}$$

The n-point green function eq. (95) is rewritten as

$$\begin{aligned}
G_n(p_1,\cdots,p_n) = \int\frac{\mathrm{d}^3p_\Psi}{(2\pi)^3}\frac{1}{2E_\Psi}\int \mathrm{d}^4x_1\mathrm{d}^4x_{r+1}\,e^{-ip_\Psi(x_1-x_{r+1})}\,G_r^A(p,p_2,\cdots,p_r)G_{n-r}^B(p,p_{r+2},\cdots,p_n)\\
\times\; e^{i(p_1+p_2+\cdots p_r)\cdot x_1}e^{i(p_{r+1}+p_{r+2}+\cdots+p_n)\cdot x_{r+1}} + \text{extra}
\end{aligned} \tag{112}$$

where I have defined $G_r^A(p,p_2,\cdots,p_r)$ and $G_{n-r}^B(p,p_{r+2},\cdots,p_n)$ as

$$G_r^A(p,p_2,\cdots,p_r) = \int \mathrm{d}^4y_2\cdots\mathrm{d}^4y_r\,e^{ip_2x_r}\cdots e^{ip_rx_r}\langle\Omega|T\{\phi(0)\phi(y_2)\cdots\phi(y_r)\}|\Psi\rangle \tag{113}$$

and

$$G_{n-r}^B(p,p_{r+1},\cdots,p_n) = \int \mathrm{d}^4y_{r+2}\cdots\mathrm{d}^4y_n\,e^{ip_{r+2}x_{r+2}}\cdots e^{ip_nx_n}\langle\Psi|T\{\phi(0)\phi(y_2)\cdots\phi(y_n)|\Omega\rangle \tag{114}$$

Also, we have

$$\int \frac{\mathrm{d}^3 p_\Psi}{(2\pi)^3} \frac{1}{2E_\Psi} e^{-ip_\Psi (x_1 - x_{r+1})}\Bigg|_{p^0_\Psi = E_\Psi} = \int_{x^0_1 > x^0_{r+1}} \frac{\mathrm{d}^4 p_\Psi}{(2\pi)^4} \frac{i}{p^2_\Psi - m^2 + i\varepsilon} e^{-ip_\Psi (x_1 - x_{r+1})} \tag{115}$$

and performing the $\mathrm{d}^4 x_1$ integral over the exponentials containing x_1 gives

$$\int \mathrm{d}^4 x_1 e^{-ip_\Psi x_1} \cdot e^{i(p_1 + p_2 + \cdots + p_r)\cdot x_1} = (2\pi)^4 \delta^{(4)}(p_\Psi - (p_1 + p_2 + \cdots + p_r)) \tag{116}$$

for $\mathrm{d}^4 x_{r+1}$ case gives

$$\int \mathrm{d}^4 x_{r+1} e^{ip_\Psi x_{r+1}} \cdot e^{i(p_{r+1} + p_{r+2} + \cdots + p_n)\cdot x_{r+1}} = (2\pi)^4 \delta^{(4)}(p_\Psi + (p_{r+1} + p_{r+2} + \cdots + p_n)) \tag{117}$$

Then,

$$\begin{aligned} G_n(p_1, \cdots, p_n) = \int \frac{\mathrm{d}^4 p_\Psi}{(2\pi)^4} \frac{i}{p^2_\Psi - m^2 + i\varepsilon} &(2\pi)^8 \delta^{(4)}(p_\Psi - (p_1 + p_2 + \cdots + p_r)) \\ &\times \delta^{(4)}(p_\Psi + (p_{r+1} + p_{r+2} + \cdots + p_n)) G^A_r(p, p_2, \cdots, p_r) G^B_{n-r}(p, p_{r+2}, \cdots, p_n) \\ &\quad + \text{extra} \end{aligned} \tag{118}$$

becomes

$$\begin{aligned} G_n(p_1, \cdots, p_n) = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + \cdots + p_n) &\frac{i}{q^2 - m^2 + i\varepsilon} G^A_r(p, p_2, \cdots p_r) G^B_{n-r}(p, p_{r+2}, \cdots, p_n) \\ &+ \text{extra} \end{aligned} \tag{119}$$

This equation says that Green's functions always have poles when on-shell intermediate particles can be produced.

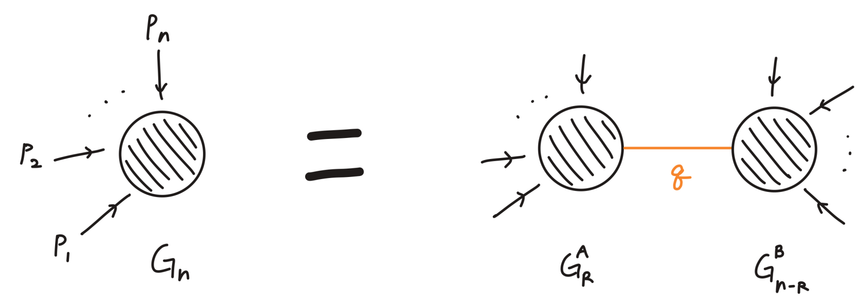

Figure 3: Pology

In driving eq. (119), the only thing we used was that the state $|\Phi\rangle$ is a one-particle state with overlap with the state with r fields $\phi_1 \cdots \phi_r$. We never needed to associate Ψ with a field in a Lagrangian. This formula does not distinguish elementary particles (those with corresponding fields in a Lagrangian) from composite particles.

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